It’s now time to introduce time. So far we have only had a static picture. Now we’ll look at the dynamics.

Keeping with the relativistic spirit, we put space and time on an equal footing. That means that if $\psi$ is a function of $x$, it is also a function of $t$.

$$\psi(x, t)$$

In quantum field theory this becomes an operator that creates or annihilate particles at a space-time point. More of that later (next semester). For now we can think of $|\psi(x, t)|^2$ as being the space-time probability density.

$$d\text{Prob}(x, t) = |\psi(x, t)|^2 dx \, dt$$

Or in three spatial dimensions

$$d\text{Prob}(\vec{x}, t) = |\psi(x, t)|^2 d^3 x \, dt = |\psi(x, t)|^2 d^4 x$$

$\psi(\vec{x}, t)$ is a Lorentz invariant.

Now need equation of motion to tell us how $\psi(x, t)$ evolves.
Some Relativity

A couple of 4-vectors

4-momentum: \[ p^\mu = (E/c, \vec{p}) \]

4-vector gradient: \[ \partial^\mu = \left( \frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right) \]

Energy-momentum relations

\[ \vec{p} = \gamma m \vec{v} = \gamma \vec{\beta} mc \quad E = \gamma mc^2 \]

where \[ \vec{\beta} = \vec{v}/c \quad \gamma = (1-v^2/c^2)^{-1/2} = (1-\beta^2)^{-1/2} \]

\[ E = \sqrt{p^2c^2 + m^2c^4} \]

\[ E^2 = p^2c^2 + m^2c^4 \]
Energy Operator

We have seen that the momentum operator is:

\[ \hat{p} = -i\hbar \frac{\partial}{\partial x} \]

or in three spatial dimensions

\[ \vec{\hat{p}} = -i\hbar \vec{\nabla} \]

In conformance to relativistic invariance, we must therefore have

\[ \hat{E} = i\hbar \frac{\partial}{\partial t} \]

\[ \hat{p}^{\mu} = i\hbar \left( \frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right) \]
Since in the macroscopic regime, wave packets must correspond to classical particles, we have:

\[ \langle E^2 \rangle = \langle p^2 \rangle c^2 + m^2 c^4 \]

\[ \int_{-\infty}^{\infty} \psi^* (\vec{x}, t) \left( -\hbar^2 \frac{\partial^2}{\partial t^2} \right) \psi (\vec{x}, t) \, d^3 x \]

\[ = \int_{-\infty}^{\infty} \psi^* (\vec{x}, t) \left( -\hbar^2 c^2 \nabla^2 + m^2 c^4 \right) \psi (\vec{x}, t) \, d^3 x \]

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \psi (\vec{x}, t) = 0 \]

\[ \left( \partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2} \right) \psi (\vec{x}, t) = 0 \]

This is the Klein-Gordon equation.
Solutions to Klein-Gordon Equation

Definite energy/momentum state solutions to the Klein-Gordon equation.

\[
\left( \partial_\mu \partial^\mu + \frac{m^2c^2}{\hbar^2} \right) \psi(\vec{x}, t) = 0
\]

\[
\psi(\vec{x}, t) = Ne^{i(p \cdot \vec{x} - Et)/\hbar} = Ne^{ip \cdot x/\hbar}
\]

Just as a state of definite momentum has a spatial probability distribution that is constant from \( x = -\infty \) to \( x = +\infty \).

\[
\psi(x) \sim e^{ipx/\hbar}
\]

A state of definite energy would have to have equal probability for the particle to be found at any time from \( t = -\infty \) to \( t = +\infty \). Since our universe had a start, the Big Bang, this isn’t possible. Also, even if the universe went back to \( t = -\infty \), the probability of finding the particle during any finite interval of time would be zero. Conclusion, a state of definite energy doesn’t physically exist.
Massless Particle

\[ m = 0 \quad \partial_\mu \partial^\mu \psi(\vec{x}, t) = 0 \]

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \psi(\vec{x}, t) = 0 \]

This is the wave equation, the equation for wave propagation on a string of velocity equal to \( c \).

In one spatial dimension:

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi(\vec{x}, t) = 0 \]

\[ \psi(x, t) \sim e^{i(px - Et)/\hbar} = e^{i(kx - \omega t)} \]

\[ \omega = \frac{E}{\hbar} \quad k = \frac{p}{\hbar} \]

\[ v = \frac{\omega}{k} = \frac{E}{p} = \frac{pc}{p} = c \]

For \( m = 0 \), all waves propagate at the velocity \( c \) independent of the frequency (energy). There is no dispersion.
Particle with Mass

\[ m \neq 0 \quad \left( \partial_{\mu} \partial^{\mu} + \frac{m^2 c^2}{\hbar^2} \right) \psi(\vec{x}, t) = 0 \]

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \psi(\vec{x}, t) = 0 \]

In one spatial dimension:

\[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{m^2 c^2}{\hbar^2} \right) \psi(\vec{x}, t) = 0 \]

\[ \psi(x, t) \sim e^{i(px - Et)/\hbar} = e^{i(kx - \omega t)} \]

\[ E^2 = p^2 c^2 + m^2 c^4 \quad \omega^2 = k^2 c^2 + m^2 c^4 \]

\[ v_{\text{phase}} = \frac{\omega}{k} = \frac{E}{p} = \gamma mc^2 \quad \gamma mv = \frac{c^2}{v} \]

Plane waves propagate at the superluminal speed \( c^2 / v \)

Dispersion:

\[ v_{\text{phase}} = \frac{\omega}{k} = c \sqrt{1 + \frac{m^2 c^2}{k^2}} \]
Group Velocity

The velocity that we calculated on the previous page is the phase velocity the velocity of a pure frequency plane wave. As we’ve seen, physical particles are described by wave packets that are a linear superposition of plane waves. The velocity of the average position of a wave packet is given by the group velocity.

\[ v_{\text{group}} = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{d}{dp}(p^2c^2 + m^2c^4)^{1/2} \]

\[ = pc^2(p^2c^2 + m^2c^4)^{-1/2} = \frac{pc^2}{E} = \frac{\gamma mvc^2}{\gamma mc^2} = v \]

So, the group velocity of the wave packet corresponds to the velocity \( v \) of a classical particle as it should.
Conserved Probability Current

\[
\left( \partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2} \right) \psi(x, t) = 0
\]

\[
\left( \partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2} \right) \psi^*(x, t) = 0
\]

\[
\partial_\mu (\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*) = 0
\]

\[
j^\mu = i (\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*)
\]

\[
\begin{cases}
  j^0 = \frac{i}{c} \left( \psi^* \frac{\partial}{\partial t} \psi - \psi \frac{\partial}{\partial t} \psi^* \right) \\
  \vec{j} = -i(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)
\end{cases}
\]

for a plane-wave: \( \psi(x, t) = N e^{-ip \cdot x/\hbar} \)

\[
j^0 = |N|^2 \frac{2E}{\hbar c} \quad \vec{j} = |N|^2 \frac{2\vec{p}}{\hbar}
\]

\[
j^\mu = |N|^2 \frac{2p^\mu}{\hbar} = (\rho c, \rho \vec{v})
\]

\[
\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0
\]