Lecture 2

Wave Functions

September 8, 2010
Measurement

It is important to understand the concept of measurement in quantum mechanics and how it differs from classical physics

• In classical physics, the position and momentum of a particle exists independently of any measurement. In quantum mechanics, the position and momentum of a particle only have physical meaning after they are measured.

• In classical physics, an ideal measurement does not affect the state of the system. In quantum mechanics, in general, a measurement changes the state of the system. It changes $\psi(x)$.

• In quantum mechanics, we can only measure the position of a particle or its momentum. We cannot measure both simultaneously.

• We will sometimes speak of an ideal measurement that precisely determines the position or precisely determines the momentum. In practice this isn’t possible, rather the position or momentum can only be determined within some finite region. In reality, a precise position or a precise momentum for a particle may not even be possible. We’ll discuss this shortly
Now for the hard one.

- A measurement has a non-local effect. It will instantaneously change the wave function everywhere even halfway across the universe. Many people don’t like this but it doesn’t matter. That’s the way things are.

For example, a particle may have a wave function that extends from here to the Moon. That means that in a measurement of its position, it has some probability to be on the Moon and some probability to be in front of me. If I measure it to be on the table in front of me, the probability of it being on the Moon instantaneously goes to zero.
Relation between $\psi(x)$ and $\tilde{\psi}(p)$

Since $\psi(x)$ and $\tilde{\psi}(p)$ each contain full information on the state of the system, we must be able to write one in terms of the other. Only possibility is:

$$
\tilde{\psi}(p) = \int_{-\infty}^{\infty} f(x, p) \psi(x) \, dx
$$

$$
\psi(x) = \int_{-\infty}^{\infty} g(x, p) \tilde{\psi}(p) \, dp
$$

We can derive what the functions $f(x, p)$ and $g(x, p)$ must be by examining how $\psi(x)$ and $\tilde{\psi}(p)$ must transform under translations in space.
Translation Properties

translation: \[ x \rightarrow x + a \]
\[ p \rightarrow p \]

\[ \psi(x) \rightarrow \psi'(x) \quad \tilde{\psi}(p) \rightarrow \tilde{\psi}'(p) \]

\[ |\psi'(x)|^2 = |\psi(x - a)|^2 \quad |\tilde{\psi}'(p)|^2 = |\tilde{\psi}(p)|^2 \]

\[ \tilde{\psi}'(p) = e^{-i\alpha(p)a} \tilde{\psi}(p) \]

assume \( \alpha(p) \) is simplest nontrivial function
\[ \alpha(p) = p/\hbar \]

for infinitesimal translation \[ x \rightarrow x + \epsilon \]

\[ \tilde{\psi}'(p) = \left( 1 - \frac{i\epsilon p}{\hbar} \right) \tilde{\psi}(p) = \tilde{\psi}(p) - \frac{i\epsilon p}{\hbar} \tilde{\psi}(p) \]

\[ \psi'(x) = \psi(x) - \epsilon \frac{d\psi(x)}{dx} \]
Expression for $g(x, p)$

$$\psi'(x) = \int_{-\infty}^{\infty} g(x, p) \tilde{\psi}'(p) \, dp$$

$$\psi(x) - \epsilon \frac{d\psi(x)}{dx} = \int_{-\infty}^{\infty} g(x, p) \tilde{\psi}(p) \, dp - \epsilon \int_{-\infty}^{\infty} g(x, p) \left( \frac{ip}{\hbar} \right) \tilde{\psi}(p) \, dp$$

$$\frac{d\psi(x)}{dx} = \int_{-\infty}^{\infty} g(x, p) \left( \frac{ip}{\hbar} \right) \tilde{\psi}(p) \, dp$$

$$\frac{d}{dx} \left( \int_{-\infty}^{\infty} g(x, p) \tilde{\psi}(p) \, dp \right) = \int_{-\infty}^{\infty} g(x, p) \left( \frac{ip}{\hbar} \right) \tilde{\psi}(p) \, dp$$

$$\int_{-\infty}^{\infty} \frac{dg(x, p)}{dx} \tilde{\psi}(p) \, dp = \int_{-\infty}^{\infty} g(x, p) \left( \frac{ip}{\hbar} \right) \tilde{\psi}(p) \, dp$$

$$\frac{dg(x, p)}{dx} = g(x, p) \left( \frac{ip}{\hbar} \right) \Rightarrow g(x, p) = Ae^{ipx/\hbar}$$
Dirac Delta Function

\[ \psi(x) = \int_{-\infty}^{\infty} g(x, p) \tilde{\psi}(p) \, dp \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, p) f(x', p) \psi(x') \, dx' \, dp \]

\[ = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} g(x, p) f(x', p) \, dp \right) \psi(x') \, dx' \]

\[ = \int_{-\infty}^{\infty} \delta(x - x') \psi(x') \, dx' \]

\[ \int_{-\infty}^{\infty} g(x, p) f(x', p) \, dp = \delta(x - x') \]

\[ \delta(x - x') = 0 \quad \text{if} \ x \neq x' \]

\[ = \infty \quad \text{if} \ x = x' \]
Properties of the Dirac Delta Function

\( \delta(x) \) is not an analytic function. It cannot be expressed as a power series.

It is a \( C^{-2} \) function. It must be integrated twice in order to get a continuous function.

One integration gives the step function \( \theta(x) \)

\[
\theta(x - a) = \int_{-\infty}^{x} \delta(x' - a) \, dx'
\]

\[
\theta(x) = 0 \quad \text{for } x < 0
\]

\[
= 1 \quad \text{for } x > 0
\]

The delta function is normalized.

\[
\int_{-\infty}^{\infty} \delta(x - a) \, dx = 1
\]
Formula for the Dirac Delta Function

\[ \frac{\sin Kx}{x} \]

peak value \( = K \)

first zero at \( x = \pi/K \)

\[ \int_{-\infty}^{\infty} \frac{\sin Kx}{x} \, dx = \pi \]

\[ \delta(x) = \lim_{K \to \infty} \frac{1}{\pi} \frac{\sin Kx}{x} \]

\[ \int_{-K}^{K} e^{ikx} \, dk = \frac{1}{i\pi} \left( e^{iKx} - e^{-iKx} \right) = 2 \frac{\sin Kx}{x} \]

\[ \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk \]
Dirac Delta Function Identities

\[
\delta(x) = \delta(-x)
\]

\[
\delta'(x) = -\delta'(-x)
\]

\[
\delta(ax) = \frac{1}{|a|} \delta(x)
\]

\[
 x\delta(x) = 0
\]

\[
f(x)\delta(x - a) = f(a) \delta(x - a)
\]

\[
\delta(f(x)) = \sum_i \frac{1}{|df/dx|_{x=a_i}} \delta(x - a_i)
\]

where \(a_i\) are the roots of \(f(x) = 0\)

\[
\int_{-\infty}^{\infty} f(x)\delta'(x) \, dx = -f'(0)
\]
Fourier Transforms

\[ g(x, p) = Ae^{ipx/\hbar} \]

\[ \int_{-\infty}^{\infty} g(x, p)f(x', p) \, dp = \delta(x - x') \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} \, dk \]
\[ = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ip(x-x')/\hbar} \, dp \]

\[ f(x, p) = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \quad g(x, p) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \]

\[ \tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x) \, dx \]

\[ \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \tilde{\psi}(p) \, dp \]