Nightmare

Find the frequency of small oscillations of the mass \( m \) in the system shown in Fig. 1. (Assume that the pulley is a solid disc of mass \( M \) and radius \( R \), and the string does not slip on the pulley).

Solution

The first thing to do is to notice that before the “small” displacement \( x \) is made, the spring is already stretched by some \( x_0 \), which can be found by considering the static equilibrium of the system. In this case,

\[
T_1 = mg
\]

and, since the pulley does not rotate in this state,

\[
T_2 = T_1
\]
Finally:

\[ T_2 = \kappa x_0 \]

and all of these equations together yield:

\[ \kappa x_0 = mg \]  \hspace{1cm} (1)

as expected. If you did not do this step, your final answer would still have remained unchanged since \( \kappa x_0 \) and \( mg \) merely contribute a constant in that case and do not affect the frequency of oscillations.

When the small displacement \( x \) is made, the system now is not in equilibrium. If we assume \( x \) was made above the equilibrium position, then the system will tend to retract back, pulling \( m \) up in the process. In this scenario, Newton’s second law for the system may be written down as, starting from the right:

\[ T_2 = \kappa (x + x_0) \]  \hspace{1cm} (2)

For the pulley, since it rotates, we have:

\[ T_2 R - T_1 R = I \alpha = \frac{1}{2} MR^2 \alpha \]  \hspace{1cm} (3)

and for the mass \( m \):

\[ T_1 - mg = ma \]  \hspace{1cm} (4)

Substituting (2) and (4) into (3), we have:

\[ \kappa (x + x_0)R - (mg + ma)R = \frac{1}{2} MR^2 \alpha \]

Upon noticing that the angular acceleration \( \alpha \) is related to the linear acceleration \( a \) as \( \alpha = a/R \), and using the equilibrium equation (1), we end up with:

\[ \kappa x - ma = \frac{1}{2} Ma \]

or:

\[ a = \frac{\kappa}{m + \frac{1}{2} M} x \]

We are essentially done... All we need to note is that \( a = -\frac{d^2 x}{dt^2} \), since \( x \) is decreasing, while the acceleration needs to be a positive quantity. This is where the negative sign of the simple harmonic motion comes from. Therefore, identifying this with

\[ \frac{d^2 x}{dt^2} = -\omega^2 x \]

we have:

\[ \omega = \sqrt{\frac{\kappa}{m + \frac{1}{2} M}} \]

as the frequency of oscillations.

A simple exercise in writing down Newton’s second law!