1 Functional Integrals for Fermions

This approach can be extended to Fermions in two formally similar but conceptually inequivalent ways:

In the first we define coherent states of Fermions by allowing to take linear combination of states with coefficients valued in a Grassmann algebra, i.e., the algebra generated by $2^N$ anticommuting objects $\{\xi_i\xi_i^*; i = 1, \ldots, N\}$. Any member of this algebra is called a Grassmann number. $\int d\xi_j \equiv \frac{\partial}{\partial\xi_j}$ anticommutates with $\xi_j$, $j \neq i$.

We define a fermion coherent state as

$$|\xi\rangle \equiv \exp\left(-\sum_{\alpha} \xi_\alpha a_\alpha |0\rangle\right) \prod_{\alpha} (1 - \xi_\alpha a_\alpha^\dagger) |0\rangle$$

We require that $\{\xi, a\} = 0$ and $(\xi a)^\dagger = a^\dagger \xi^*$. Then it is clear that

$$a_\alpha |\xi\rangle = \xi_\alpha |\xi\rangle$$
$$\langle \xi | a_\alpha^\dagger = \langle \xi | \xi_\alpha^*$$

To understand the properties of the Fermionic coherent states consider first a single Fermion degree of freedom $\{a, a^\dagger\}$. Define

$$|\xi\rangle \equiv e^{-\xi a^\dagger} |0\rangle$$

$\{\xi, a\} = 0$ and $(\xi a)^\dagger = a^\dagger \xi^*$. The overcompleteness relations are easily worked out

$$\int d\xi d\xi^* \langle \xi | \langle \xi | = \frac{\partial}{\partial\xi^*} \frac{\partial}{\partial\xi} (|0\rangle - \xi a^\dagger |0\rangle)(\langle 0| - \langle 0| a\xi^*) (1 - \xi^* \xi)$$
$$= |0\rangle \langle 0| + a^\dagger |0\rangle \langle 0| a = I$$

Notice that $\xi$ anticommutates with vectors in the Hilbert space containing an odd number of Fermions in which case

$$\xi |\psi\rangle = - |\psi\rangle \xi$$
$$\langle \eta |\xi\rangle = (0 - \langle 0| a\eta^*)(|0\rangle - \langle 0| a\xi^*)$$
$$= 1 + \eta^* \xi = e^{\eta^* \xi}$$

Finally, notice that if $|\psi\rangle$ are vectors in the Fock space with a definite number of Fermions then $\langle \xi |\psi\rangle$ is a grassmann variable which contains an odd number of $\eta$, if $|\psi\rangle$ has an odd number of Fermions and contains an even number of $\eta$, if $|\psi\rangle$ has an even number of Fermions. Hence:

$$\langle \xi |\psi\rangle \langle \psi_j |\xi\rangle = \langle \psi_j |\xi\rangle \langle \xi |\psi\rangle (-1)^F$$
$$= \langle \psi_j |\xi - \xi\rangle = \langle \psi_j |\xi\rangle - \langle \psi_j |\xi\rangle$$
This gives an expression for the trace of an operator $A$ which acts on the Fock space and preserves the number of particles.

$$\text{Tr} A = \sum_n \langle n | A | n \rangle = \sum_n \int \prod_\alpha d\xi_\alpha d\xi_\alpha e^{-\sum_\alpha \xi_\alpha \xi_\alpha} \langle n | \xi | n \rangle \langle \xi | A | n \rangle$$

$$= \int \prod_\alpha d\xi_\alpha d\xi_\alpha e^{-\sum_\alpha \xi_\alpha \xi_\alpha} \sum_n \langle -\xi | A | n \rangle \langle n | \xi \rangle$$

Equipped with these techniques we can write functional integral expression for Fermi systems in complete analogy with the Bosonic theory.

$$U(\tau, \tau') = \prod_{i=0}^{N-1} U(\tau_{i+1}, \tau_i)$$

$\tau_0 = \tau', \tau_N = \tau$. We express

$$U(\tau_{i+1}, \tau_i) = \exp -H \left( \frac{\tau_{i+1} + \tau_i}{2} \right) \Delta \tau$$

and insert at each point a resolution of the identity to find

$$\langle \eta_N | U(\tau, \tau') | \eta_0 \rangle = \prod_{i=1}^{N-1} \int d\eta_i^* d\eta_i e^{-\eta_i^* \eta_i} \langle \eta_{i+1} | U(\tau_{i+1}, \tau_i) | \eta_i \rangle \langle \eta_1 | U(\tau_1, \tau_0) | \eta_0 \rangle$$

Using the expression for the trace of an operator derived earlier one finds

$$\text{Tr} U(\tau, \tau') = \int d\eta_0^* d\eta_0 e^{-\eta_0^* \eta_0} \langle \eta_0 | U(\tau, \tau') | \eta_0 \rangle$$

If $U \simeq e^{-H \left( \frac{\tau_{i+1} + \tau_i}{2} \right) \Delta \tau}$ and $H_c(\eta^*, \eta)$ with $\eta^*$ and $\eta$ and replaced by $a^\dagger$ and $a$ give the normal ordered form of $H$ we find

$$\text{Tr} U(\tau, \tau') = \prod_{i=0}^{N-1} \int d\eta_i^* d\eta_i e^{-\sum_{i=0}^{N-1} \eta_i^* \eta_i} \sum_{i=0}^{N-1} \eta_i^* \eta_i^\dagger e^{-H_c(\eta_i^\dagger + \eta_i)} \Delta \tau$$

$$(\eta_N^\dagger \equiv -\eta_0^\dagger).$$

Notice that if $H_c$ contained some explicit time dependence due to the source or an auxiliary field, it should be evaluated at time $\tau = \frac{\tau_{i+1} + \tau_i}{2}$.

Proceeding heuristically we could define

$$\text{Tr} e^{-\beta H} = \int D\eta^* D\eta e^{-\left( \eta^* \frac{\partial}{\partial \eta} + H_c(\eta^*, \eta) \right)}$$

where the operator $\frac{\partial}{\partial \eta}$ is defined on the space of functions obeying $\eta(\beta) \equiv -\eta(0)$. 

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2 The Standard Gaussian Integral:

\[ \int \prod_i d\eta_i^* d\eta_i e^{-\eta_i^* A \eta_i + \eta_j^* j^* \eta_j} = \int \prod_i d\eta_i^* d\eta_i e^{-\eta_i^* A \eta_i} e^{j^* A^{-1} j} \]

using the transformation

\[
\eta \rightarrow \eta + A^{-1} j \\
\eta^* \rightarrow \eta^* + j^* A^{-1}
\]

Thus

\[ \int \prod_i d\eta_i^* d\eta_i e^{-\eta_i^* A \eta_i + \eta_j^* j^* \eta_j} \simeq (\det A)e^{j^* A^{-1} j} \]

expanding to second order.

\[
\eta_i^* j_i j_k^* \eta_k = j_k^* A^{-1}_{kl} j_l \\
\langle \eta_k | \eta_i^* \rangle = A^{-1}_{kl}
\]