Problem set 1 (version Wed Sep 20) (Due on Thursday September 28th)

1. Consider the simple harmonic oscillator

\[ \hat{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \]

Write it in “second quantized” form, by expressing x and p in terms of creation and annihilation operators.

\[ \hat{H} = \hbar\omega (a^\dagger a + \frac{1}{2}) \].

(i) Derive the equation of motion for \( a \) and show that

\[ a(t) = a(0)e^{-i\omega t}. \]

Repeat this for \( a^\dagger \).

(ii) Using the relationship

\[ a^\dagger = \frac{1}{\sqrt{2\hbar}} \left[ \frac{\hat{p}}{\sqrt{m\omega}} + i\hat{x}\sqrt{m\omega} \right] \]

use the results of (i) to show that

\[ \hat{p}(t) = \hat{p}(0)\cos \omega t - m\omega \hat{x}(0)\sin \omega t \]
\[ \hat{x}(t) = \hat{x}(0)\cos \omega t + \frac{\hat{p}(0)}{m\omega} \sin \omega t. \]

(iii) The above operator expressions appear identical to the classical equations of motion.

Why?

2. Show that the Heisenberg spin operator S can be written in second quantized form as

\[ S_x = \frac{1}{2}(c_1^\dagger c_1 + c_1^\dagger c_1^\dagger), \]
\[ S_y = \frac{i}{2}(c_1^\dagger c_1 - c_1^\dagger c_1^\dagger), \]
\[ S_z = \frac{1}{2}(c_1^\dagger c_1 - c_1^\dagger c_1^\dagger). \]
Check the commutation relations!

3. Consider a system of fermions created by the field $\psi \dagger (\vec{r})$ interacting under the Yukawa potential

$$V(r) = \frac{Ae^{-\lambda r}}{4\pi r}.$$ 

(i) Write the Hamiltonian in second quantized form, using the position basis.

(ii) Write the Hamiltonian in second quantized notation in the momentum basis, where

$$c_k^\dagger = \int d^3r \psi^\dagger(\vec{r})e^{i\vec{k}.\vec{r}}.$$ 

You will find it helpful to derive the Fourier representation

$$V(r) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}.\vec{r}} \frac{A}{(q^2 + \lambda^2)}.$$ 

4. For the spin 1/2 fermions considered in question 3, prove the “Hubbard Stratonovich” identity

$$e^{-a(n_1 - 1/2)(n_1 - 1/2)} = \frac{1}{2} e^{-a/4} \sum_{\sigma=\pm 1} e^{a\sigma(n_1 - n_1)}$$

where $\cosh a = e^{a/2}$ (Hint: consider the matrix representation of the operator on the left hand side in the four dimensional Fock space of the up and down electrons.) We shall later see how, considering $a \sim \delta t$ to be an infinitesimal time interval, we can use this relation to treat interacting spins as spins moving in a fluctuating Ising magnetic field.

6. Let $a^\dagger_\alpha, a_\alpha$ be Fermion creation and annihilation operators, and assume

$$H_o = \sum_\alpha \epsilon_\alpha a^\dagger_\alpha a_\alpha.$$ 

(i) Compute $Tr[e^{\beta H_o}a^\dagger_\alpha a_\alpha]$

(ii) Compute $Tr[e^{-\beta H_o}a_\alpha a^\dagger_\alpha]$.
(iii) Compute $Tr[e^{-\beta H_o}a_\alpha a_\alpha^\dagger]$. Using the identity $e^{-\beta H_o}a_\alpha^\dagger a_\alpha = -T \frac{\partial}{\partial \epsilon_\alpha} e^{-\beta H_o}$ check your answer to (i).

(iv) Confirm that the expectation
\[ \langle n_\alpha \rangle = Z^{-1} Tr[e^{-\beta H_o} a_\alpha^\dagger a_\alpha] = f(\epsilon_\alpha) \]
where $f(x)$ is the Fermi function.

(v) Repeat the above procedure assuming that $a_\alpha^\dagger$ and $a_\alpha$ are boson operators. What are the restrictions on the values of $\epsilon_\alpha$ and what happens when one of these energies $\epsilon_\alpha$ becomes zero?

(vi) Calculate
\[ \begin{pmatrix} a_\alpha^\dagger(\tau) \\ a_\alpha(\tau) \end{pmatrix} = e^{\tau H_o} \begin{pmatrix} a_\alpha^\dagger \\ a_\alpha \end{pmatrix} e^{-\tau H_o} \]