Physics 511: Problem Set 1

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1. Classical matrix groups

a.) The orthogonal groups are defined by:

\[ O(n, \kappa) := \{ A \in M_n(\kappa) : AA^tr = 1 \} \]
\[ SO(n, \kappa) := \{ A \in O(n, \kappa) : \det A = 1 \} \]  

(1.1)  \textbf{eq:orthgroup}

where \( M_n(\kappa) \) is the set of \( n \times n \) matrices over the field \( \kappa \). (You can take \( \kappa = \mathbb{R}, \mathbb{C} \).) Show that they are groups.

b.) Similarly, the symplectic groups are defined by:

\[ Sp(2n, \kappa) := \{ A \in M_n(\kappa) | A^tr JA = J \} \]

(1.2)  \textbf{eq:SymplecticGroup}

where

\[ J = \begin{pmatrix} 0 & 1_{n \times n} \\ -1_{n \times n} & 0 \end{pmatrix} \in M_{2n}(\mathbb{R}) \]  

(1.3)  \textbf{eq:symplecticform}

Show that \( Sp(2n, \kappa) \) is a group.
c.) Show that if $A$ is a symplectic matrix then $A^{tr}$ is a symplectic matrix.

d.) Decompose a symplectic matrix into $n \times n$ blocks:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(1.4)

where $a, b, c, d \in M_n(\kappa)$. Write necessary and sufficient conditions for $A$ to be a symplectic matrix.

2. Symplectic groups and canonical transformations

Let $q^i, p_i \ i = 1, \ldots, n$ be coordinates and momenta for a classical mechanical system.

The Poisson bracket of two functions $f(q^1, \ldots, q^n, p_1, \ldots, p_n)$, $g(q^1, \ldots, q^n, p_1, \ldots, p_n)$ is defined to be

$$\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$

(2.1)

a.) Show that

$$\{q^i, q^j\} = \{p_i, p_j\} = 0 \quad \{q^i, p_j\} = \delta^i_j$$

(2.2)

Suppose we define new coordinates and momenta $Q^i, P_i$ to be linear combinations of the old:

$$\begin{pmatrix} Q^1 \\ \vdots \\ Q^n \\ P_1 \\ \vdots \\ P_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1,2n} \\ \vdots & \ddots & \vdots \\ a_{2n,1} & \cdots & a_{2n,2n} \end{pmatrix} \begin{pmatrix} q^1 \\ \vdots \\ q^n \\ p_1 \\ \vdots \\ p_n \end{pmatrix}$$

(2.3)

where $A = (a_{ij})$ is a constant $2n \times 2n$ matrix.

b.) Show that

$$\{Q^i, Q^j\} = \{P_i, P_j\} = 0 \quad \{Q^i, P_j\} = \delta^i_j$$

(2.4)

if and only if $A$ is a symplectic matrix.

3. Decomposing the reverse shuffle

Consider the permutation which takes $1, 2, \ldots, n$ to $n, n-1, \ldots, 1$.

a.) Write the cycle decomposition.

b.) Write a decomposition of this permutation in terms of the elementary generators $\sigma_i = (i, i+1)$. (See the exercise in Section 4.3.)

4. Subgroups of $A_4$

Write down all the subgroups of $A_4$. 

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5. Words in a finite Heisenberg group

As in the notes define $P, Q$ be $N \times N$ “clock” and “shift” matrices:

$$P_{i,j} = \delta_{j=i+1 \text{mod} N}$$  \hfill (5.1) \text{eq:ShiftMatrix}

$$Q_{i,j} = \delta_{i,j} \omega^j$$  \hfill (5.2) \text{eq:ClockMatrix}

a.) Show that the word

$$P^{n_1} Q^{m_1} P^{n_2} Q^{m_2} \cdots P^{n_k} Q^{m_k}$$

where $n_i, m_i \in \mathbb{Z}$ can be written as $\xi P^x Q^y$ where $x, y \in \mathbb{Z}$ and $\xi$ is an $N^{th}$ root of unity. Express $x, y, \xi$ in terms of $n_i, m_i$.

b.) Let $\text{Heis}_N$ the the group generated by $P, Q$. Show that there is an exact sequence

$$1 \to \mathbb{Z}_N \to \text{Heis}_N \to \mathbb{Z}_N \times \mathbb{Z}_N \to 1$$ \hfill (5.4)

6. $p$-groups

a.) Show that $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

b.) Show more generally that if $p$ is prime $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^{n-m}} \oplus \mathbb{Z}_{p^m}$ are not isomorphic if $0 < m < n$.

c.) How many nonisomorphic abelian groups have order $p^n$?