1. Let the angle the ray makes in the ith slab be $\theta_i$. Then from Snell’s Law $1.6 \sin 30^\circ = 1.6 \sin \theta_1 = 1.4 \sin \theta_2 = 1.2 \sin \theta_3 = 1.0 \sin \theta_4$,
giving $\theta_1 = 34.23^\circ$, $\theta_2 = 40.00^\circ$, $\theta_3 = 48.59^\circ$, and $\theta_4 = 64.16^\circ$. To plot the ray it is useful to find the x-coordinates $x_i$ where the ray exits the ith slab, assuming the ray enters the 1st slab at $x_0 = 0$, and using $x_i = x_{i-1} + 1cm \times \sin \theta_i$. This gives $x_1 = 0.680cm$, $x_2 = 1.520cm$, $x_3 = 2.653cm$, $x_4 = 4.718cm$. (The corresponding $y_i’s$ are of course just $i \times 1cm$.) The path of the ray is then just the broken straight line passing through these points, as shown in the Fig. for Prob. 2.

2 (a) A small segment of the ray is the hypotenuse of a right triangle with vertical side $dy$ and horizontal side $dx$, The ray makes an angle $\theta$ with the vertical, and $\tan \theta = dx/dy$ or, equivalently, $\sin \theta = dx/\sqrt{dx^2 + dy^2} = 1/\sqrt{1 + (dy/dx)^2}$, so that Snell’s Law in this case is just $n(y)/\sqrt{1 + (dy/dx)^2} = constant = n(0) \sin \theta(0)$.

(b) One can re-write the differential equation as

$$(n/n_0)^2 = \sin \theta_0^2 + (\sin \theta_0 dy/dx)^2.$$ 

Taking the derivative of the given expression for $y$ the differential equation requires that

$$[(1 + \cos \theta_0)e^{-u} + [(1 - \cos \theta_0)e^u]^2 = \sin \theta_0^2 + [(1 + \cos \theta_0)e^{-u} - [(1 - \cos \theta_0)e^u]^2,$$

where $u = \alpha x/ \sin \theta_0$. When the square brackets are squared the squared terms on the two sides of the equations cancel. The “crossed” terms are identical except for the sign, and the $\sin \theta_0^2$ term on the right can be shown to make up the difference using $(1+\cos \theta_0)(1-\cos \theta_0) = \sin \theta_0^2$. The given expression for $y$ therefore satisfies Snell’s law , is zero when $x = 0$, and at $x = 0$ $dy/dx = 1/\tan \theta_0$ as required by the initial conditions.

(c) The linear fit to the indices of refraction requires $n_0 = 1.7$ and $\alpha = 2/17$. Assume that the slope of the curve should match $\tan \theta_1$ at $y = 0.5cm$ Snell’s
Law then requires that $\sin \theta_0 = 9/17$. The function $y(x)$ calculated with these values of $\alpha$ and $\sin \theta_0$ is plotted in the accompanying figure, along with the broken straight lines from question 1.

3. (a) From the formula above Eqn. 4.81

$$\omega_P = \sqrt{Nq_e^2/\epsilon_0 m_e}$$

$$= \sqrt{1.5 \times 10^{28}/m^3 \times (1.6 \times 10^{-19}C)^2/(8.85 \times 10^{-12}C^2/nm^2 \times 9.11 \times 10^{-31}kg)}$$

$$= 6.90 \times 10^{15} \text{rad/s}.$$ 

(b) A wavelength of $\lambda = 1 \times 10^{-6}m$ corresponds to an angular frequency $\omega = 2\pi c/\lambda = 2\pi 3 \times 10^6 m/s/1 \times 10^{-6} m = 1.885 \times 10^{15} \text{rad/s}$. The index of refraction squared is then

$$n^2 = 1 - (\omega_P/\omega)^2 = 1 - (6.90/1.886)^2 = -12.4.$$ 

The square root of $-1$ is $i$, giving $n_R = 0$ and $n_I = \sqrt{12.4} = 3.52$. Eqn. 4.83 then gives

$$R = \frac{1 + (3.52)^2}{1 + (3.53)^2} = 1.0.$$ 

In other words, if the index of refraction is a purely imaginary number, without a real part, all of the incident light is reflected. (This would not be exactly true if one did not ignore the $\gamma_e$ term in the denominator in Eqn. 4.79.)

4. From Eqn. 4.73 the amplitude of the electric field in the air falls off as $\exp(-\beta y)$ where

$$\beta = k_t\sqrt{(\sin \theta_t/n_{ti})^2 - 1}.$$ 

In this expression $t$ indicates the "transmitted ray", in the air, $k_t = 2\pi/\lambda$ is the propagation number in the air, and $n_{ti} = n_i/n_i = 1/1.6$, while $\theta_t = 45^\circ$. This gives $\beta = 6.1 \times 10^6$ m. Since the irradiance is proportional to $|E|^2$, it falls off as $\exp(-2\beta y)$, and the irradiance $1\mu = 1 \times 10^{-6} m$ above the surface is only $\exp[-2(6.1 \times 10^6/m)1 \times 10^{-6}] = 5 \times 10^{-6}$ times that at the surface where $y = 0$. 

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