1. To find the Clebsch Gordan coefficients, we start with the state formed by combining parallel spins, then we apply the lowering operator successively. In the following solutions, the Clebsh-Gordan coefficients can be read off from the corresponding expansion of the angular momentum states.

When we make the combination \( \frac{1}{2} \otimes \frac{3}{2} \) we generate a \( j = 2 \) and a \( j = 1 \) state. The total number of states is \( 2 \times 4 = 8 = 5 + 3 \), so the \( j = 1 \) and \( j = 2 \) state exhaust the total number of states.

\[
\frac{1}{2} \otimes \frac{3}{2} = 2 \oplus 1
\]  

(1)

The state with maximum \( m_z \) corresponding formed from \( j = \frac{1}{2} \) and \( j = \frac{3}{2} \) is

\[
|2, 2\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle |\frac{1}{2}, \frac{1}{2}\rangle
\]  

(2)

If we apply the lowering operator to this state we obtain

\[
J_-|2, 2\rangle = 2|2, 1\rangle
\]  

(3)

\[
J_-|\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\rangle = \frac{3}{2}\left(J_-|\frac{1}{2}, \frac{1}{2}\rangle + \langle J_-|\frac{3}{2}, \frac{3}{2}\rangle \frac{1}{2}, \frac{1}{2}\rangle
\]  

(4)

\[
= \frac{3}{2}\left|\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{3}\left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, \frac{1}{2}\rangle
\]  

(5)

so that

\[
|2, 1\rangle = \frac{1}{2}\left|\frac{3}{2}, \frac{3}{2}\rangle \frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{3}\left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, \frac{1}{2}\rangle
\]  

(6)

Repeating this process, we obtain

\[
|2, 2\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle |\frac{1}{2}, \frac{1}{2}\rangle
\]  

(7)

\[
|2, 1\rangle = \frac{1}{2}\left|\frac{3}{2}, \frac{3}{2}\rangle \frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{3}\left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, \frac{1}{2}\rangle
\]  

(8)

\[
|2, 0\rangle = \frac{1}{\sqrt{2}}\left[\left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, -\frac{1}{2}\rangle + \left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, \frac{1}{2}\rangle
\]  

(9)

\[
|2, -1\rangle = \frac{1}{2}\left[\sqrt{3}\left|\frac{3}{2}, -\frac{1}{2}\rangle \frac{1}{2}, -\frac{1}{2}\rangle + \left|\frac{3}{2}, -\frac{1}{2}\rangle \frac{1}{2}, \frac{1}{2}\rangle
\]  

(10)

\[
|2, -2\rangle = \left| \frac{3}{2}, \frac{3}{2} \right\rangle |\frac{1}{2}, -\frac{1}{2}\rangle
\]  

(11)

Now, to obtain the spin-1 states, we write

\[
|1, 1\rangle = \alpha\left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, \frac{1}{2}\rangle + \beta\left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, -\frac{1}{2}\rangle
\]  

(12)

In order that this state be orthogonal to \( |2, 1\rangle \), we choose

\[
|1, 1\rangle = \frac{1}{2}\left[\sqrt{3}\left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, -\frac{1}{2}\rangle - \left|\frac{3}{2}, \frac{1}{2}\rangle \frac{1}{2}, \frac{1}{2}\rangle
\]  

(13)
Now, successive application of the lowering operator yields

\begin{align*}
|1, 0\rangle &= \frac{1}{\sqrt{2}} \left( \frac{3}{2} \hat{\sigma}^+ |\frac{1}{2}, -\frac{1}{2}\rangle - \frac{3}{2} |\frac{1}{2}, \frac{1}{2}\rangle \right) \quad (14) \\
|1, -1\rangle &= -\frac{\sqrt{3}}{2} \left( \frac{3}{2} \hat{\sigma}^- |\frac{1}{2}, -\frac{1}{2}\rangle + \frac{3}{2} |\frac{1}{2}, \frac{1}{2}\rangle \right) - \frac{1}{2} |\frac{1}{2}, -\frac{1}{2}\rangle \quad (15)
\end{align*}

2. Sakurai, problem 3.18. The probability that the rotated state \( D(R)|l = 2, m = 0\rangle \) has \( J_z = m' \) is given by

\[ p(m') = |\langle l = 2, m | D(R) | l = 2, m = 0 \rangle|^2 = |D^{(2)}_{m'0}(R)|^2 \]

But since \( D^{(2)}_{m'0} = \sqrt{\frac{4\pi}{5}} Y^*_{2m'}(R) \), we obtain

\[ p(m') = \frac{4\pi}{5} |Y^*_{2m'}(\theta, \phi)|^2 \]

Writing this out explicitly, we obtain

\begin{align*}
p(\pm 2) &= \frac{3}{8} \sin^4 \theta, \\
p(\pm 1) &= \frac{3}{2} \sin^2 \theta \cos^2 \theta, \\
p(0) &= \frac{1}{4} (3 \cos^2 \theta - 1).
\end{align*}

3. Sakurai, problem 3.24. The singlet wavefunction of the two spins can be written

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|\hat{n}, +\rangle_1 |\hat{n}, -\rangle_2 - |\hat{n}, -\rangle_1 |\hat{n}, +\rangle_2) \]

where \( \hat{n} \) is an arbitrary observation axis.

(a) When observer B makes no measurement, the state remains in a linear superposition of the two states \( |z, \pm\rangle_1 |z, \mp\rangle_2 \) with an amplitude of magnitude \( \frac{1}{\sqrt{2}} \) to be either state. The probability for A to obtain \( s_{1z} = \hbar/2 \) is thus precisely \( \frac{1}{2} \). This holds for any quantization axis, so the probability for measuring \( s_{1x} = \hbar/2 \) is also precisely \( \frac{1}{2} \).

(b) If B measures the spin of the particle to be \( s_{2z} = \hbar/2 \), this projects the system into the state \( |z, -\rangle_1 |z, +\rangle_2 \), so that A must measure \( s_{1z} = -\hbar/2 \). If A measures \( s_{1x} \), then since

\[ |z, -\rangle_1 |z, +\rangle_2 = \frac{1}{\sqrt{2}} (|x, +\rangle_1 |z, +\rangle_2 + |x, -\rangle_1 |z, +\rangle_2) \]

there is an equal chance that A will measure \( s_{1x} = \pm \hbar/2 \).

4. (a) Now using the Wigner-Eckart theorem, we can write

\[ Q_{ab} = e^{\langle jm|x_a x_b - \frac{1}{3} r^2|jm\rangle} = \frac{\lambda}{\hbar^2} \langle jm|J_a J_b - \frac{1}{3} J^2|jm\rangle \]

Now if we evaluate \( \langle jj|Q_{33}|jj\rangle \), we obtain

\[ \langle jj|Q_{33}|jj\rangle = \frac{e}{3} \langle jj|z^2 - r^2|jj\rangle = 3Q \]
on the other hand, from the r.h.s. of the first expression, we obtain

\[ \lambda \langle jj | J_z^2 - \frac{1}{3} J^2 | jj \rangle = \lambda \frac{(2j-1)}{3} \tag{25} \]

Comparing these two results, we obtain

\[ \lambda = \frac{Q}{2(2j-1)} \tag{26} \]

(b) From the previous part of the question, we have

\[ e \langle jm | x^2 - y^2 | j'm' \rangle = \frac{Q}{\hbar^2 j(2j-1)} \langle jm | J_z^2 - J_y^2 | j'm' \rangle \]

\[ = \frac{Q}{\hbar^2 2j(2j-1)} \langle jm | J_+^2 - J_-^2 | j'm' \rangle \tag{27} \]

We can write this in the form

\[ e \langle jm | x^2 - y^2 | j'm' \rangle = \frac{Q}{\hbar^2 2j(2j-1)} \left[ \delta_{m,m'-2} \langle jm | J_z^2 | jm + 2 \rangle + \delta_{m,m'+2} \langle jm | J_z^2 | jm - 2 \rangle \right] \tag{28} \]

For the specific case of \( j = 1 \), we have

\[ \langle 11 | J_+ 2 | 1, -1 \rangle = \langle 1 -1 | J_+ 2 | 1, +1 \rangle = 2\hbar^2 \tag{29} \]

so that

\[ e \langle 1m | x^2 - y^2 | 1m' \rangle = Q \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \tag{30} \]