1. (a) Since $U|a^r⟩ = |b^r⟩ = \sum_s |a^s⟩ U_{sr}$, by writing the transformation as

$$(U|+, U|−)) = (|+, |−)) \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix}$$

we can read off the matrix elements of $U$ to be

$$[\hat{U}]_{sr} = \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix}.$$ 

(b) Under this transformation,

$$|y; ±⟩ ≡ \left(\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}\right) |y⟩ \rightarrow \left(\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}\right) \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix} \left(\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}\right) ≡ e^{\mp i\theta/2} |y; ±⟩,$$

so that

$$\hat{y}|y; ±⟩ = e^{\pm i\theta/2} |y; ±⟩.$$ 

(c) Since $H = −eB/mS_y$, $H|y; ±⟩ = ±\hbar\omega_c |y; ±⟩$, where $\omega_c = |e|B/m$, so that the time evolution of these states is given by

$$|y; ±⟩ \rightarrow e^{-i\hat{H}t/\hbar} |y; ±⟩ = e^{-i\omega_c t/2} |y; ±⟩,$$

permitting us to identify $\theta = \omega_c t$.

(d) The precession angle of the spin is given by $\theta = \omega_c t$. If $θ = 90° \equiv \pi/2$, then the time to rotate through $90°$ is

$$t = \left(\frac{\pi m}{2eB}\right) = \left(\frac{\pi \times 9.1 \times 10^{-31} \text{kg}}{2 \times 1.6 \times 10^{-19} \text{C} \times 1 \text{Tesla}}\right) = 8.9 \times 10^{-12} \text{s}.$$ 

2. Since $\psi(x) = \delta(x − x_0)$, it follows that the momentum space wavefunction is

$$\phi(p) = \langle p|\psi⟩ = \int_{−\infty}^{\infty} dx ⟨p|x⟩ ⟨x|\psi⟩ = \int_{−\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \delta(x − x_0) = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx_0/\hbar}.$$ 

(a) The time-dependent momentum space wavefunction is then given by

$$\phi(p, t) = ⟨p|e^{-i\hat{H}t/\hbar}|ψ⟩ = e^{-i\frac{p^2 x_0^2}{2m\hbar}} (p|ψ⟩) = \frac{1}{\sqrt{2\pi\hbar}} e^{-i(\frac{px_0 + x_0^2}{2m})/\hbar}. $$
(b) Transforming back to real space, we have

$$\psi(x, t) = \int_{-\infty}^{\infty} dp |x| \phi(p, t) = \int_{-\infty}^{\infty} dp e^{i (p(x-x_o) - \frac{p^2}{2m})} \phi(p, t)$$

Using the result

$$\int_{-\infty}^{\infty} dpe^{-\frac{1}{2}ap^2 + bp} = \sqrt{\frac{2\pi}{a}} \exp \left[ \frac{b^2}{2a} \right],$$

putting $a = \frac{it}{m\hbar}$ and $b = i(x-x_o)\hbar$, we obtain

$$\text{Amplitude}(x_o \rightarrow x, \Delta t) \equiv \psi(x, \Delta t) = \sqrt{\frac{m}{i\hbar \Delta t}} \exp \left[ \frac{iS}{\hbar} \right]$$

where

$$S = \frac{m}{2} \left( \frac{x-x_o}{\Delta t} \right)^2 \Delta t$$

is the classical action $S = \int_0^t dt' K.E.(t')$ for a free particle travelling from $x_o$ to $x$.

3. (a) The Hamiltonian of the simple Harmonic oscillator is

$$H = \hbar \omega [a^\dagger a + \frac{1}{2}]$$

where $a$ and $a^\dagger$ satisfy the algebra $[a, a^\dagger] = 1$. Physically, $a^\dagger$ creates a single “phonon” of energy $\hbar \omega$. The quantity $N = a^\dagger a$ is the number operator, which satisfies $\{N, a\} = [a^\dagger, a] = -a$, so that $[a, H] = -\hbar \omega [N, a] = \hbar \omega a$ and the Heisenberg equation of motion for $a(t)$ is

$$\frac{da(t)}{dt} = \frac{1}{i\hbar}[a(t), H] = -i\omega a(t)$$

which we can integrate to obtain $a(t) = e^{-i\omega t} a$.

(b) The n-th excited state $|n\rangle$ is obtained by acting on the ground-state $n$ times with the creation operator $a^\dagger$,

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$$

where the pre-factor is introduced to normalize the state.

(c) We can write $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \langle a|)$. Now the position operator $x$ can be written as

$$x = \Delta x [a + a^\dagger], \quad \Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$
In the Heisenberg representation, this becomes

\[ x(t) = \Delta x [a(t) + a^\dagger(t)] = \Delta x [ae^{-i\omega t} + a^\dagger e^{i\omega t}] \] (17)

To calculate the time dependent expectation value of position, we simply calculate the expectation value of \( x(t) \) in the state \( |\psi\rangle \), which is

\[ \langle x(t) \rangle = \langle \psi | \hat{x}(t) | \psi \rangle = \frac{\Delta x}{2} \langle 0 | + \langle 1 | [ae^{-i\omega t} + a^\dagger e^{i\omega t}] (|0\rangle + |1\rangle) \] (18)

Now only the cross-terms \( \langle 0 | a | 1 \rangle = \langle 1 | a^\dagger | 0 \rangle = 1 \) survive, so that

\[ \langle x(t) \rangle = \Delta x \cos(\omega t) = \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t) \] (19)

so in the mixed state \( |\psi\rangle = \frac{1}{\sqrt{2}} [|0\rangle + |1\rangle] \) the expectation value of the position operator oscillates like a cosine wave.

(d) The experimentalist’s results are consistent with the absorption of an odd number of photons, with frequency \( \omega \). This will then put the system in the \( n \)-th excited state. But if \( n \) is odd, the wavefunction of the system is an odd-function of position, vanishing at the origin, so that in this excited state, the electron is never found at the origin. We say that this excited state is “odd-parity” because it is odd under the reflection operator. Physically, the photon is an odd-parity particle, and this is why the absorption of odd number of photons leads to an odd-parity electron state.