Numerical Calculation

In this problem you will increase the fidelity of your calculation of the mass-radius relation for white dwarfs by allowing for the equation of state to change with radius through the star. A white dwarf is simple in some respects. It does not generate any energy from nuclear reactions, so its structure is determined by the equations of hydrostatic equilibrium (HSE)

\[ \frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \]

and mass conservation

\[ \frac{dM}{dr} = 4\pi r^2 \rho \]

as well as an equation of state. For this we will use the formula for complete degeneracy:

\[ P_e = \frac{\pi m_e^4 c^5}{3h^3} f(x) \]

\[ \rho = \mu_e \frac{8\pi m_p m_e^3 c^3}{3h^3} x^3 \]

\[ f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x. \]

Recall that we included the equation of state into the equation of hydrostatic equilibrium to obtain the following result for the gradient of density through the white dwarf

\[ \frac{d\rho}{dr} = -\frac{9Gh^3}{8\pi m_e^4 c^5} \frac{M_r \rho^2 \sqrt{1 + x^2}}{r^2 x^5} \]

Part of this exercise is to get you to learn about how to solve a differential equation on the computer, so you are not allowed to use a canned differential equation solver in mathematica, maple, or any other symbolic algebra program. Instead use a software package, such as, java, Fortran, c, c++, IDL, etc. that will allow you to do the calculation as outlined below. Be sure to indicate what program you are using.

First define a set of linearly spaced radial steps that span a large enough range to cover the full interior of the star. (You may have to experiment here.) Define corresponding arrays for the density and enclosed mass. (Note: the step size you use for the radius, \( \Delta r \), is very important for the accuracy of your solution.)

For fixed composition (here assume \( \mu_e = 2 \)), the solution depends on only one parameter, which we will take to be the central density. You will need to use several different values for the central density as initial conditions to map out the M-R relationship. Getting into the correct ball park of central density values may take some experimentation.

Once you have the central density the first thing to do is to calculate the density gradient by evaluating the RHS of the last equation above. You have the ratio \( M_r/r^2 \),
which goes to 0 at the center. (Mass goes at $r^3$ as $r$ goes to 0, which dominates the $r^2$ term in the denominator.)

Now that you have the density gradient, you can evaluate the density at the next radial grid point as $\rho_1 = \rho_0 + \Delta r * \frac{d\rho}{dr}(r_0)$. With the density value at the next grid point, you can start the process over again. In general you want to iteratively solve these two difference equations:

$$\rho_{i+1} = \rho_i - \frac{9Gh^3}{8\pi m_e^4 c^5} \frac{M_i \rho_i^2 \sqrt{1 + x_i^2}}{r_i^2 x_i^5} * \Delta r$$

and

$$M_i = M_{i-1} + 4\pi r_i^2 \rho_i * \Delta r$$

in conjunction with the relationship between $x$ and $\rho$ given above.

Continue evaluating the density until it drops to zero (note that it may go negative). The radius where this happens defines the radius of the white dwarf for that specific value of $\rho_c$.

Here’s what you need to turn in:
1. Printout of your code.
2. Table of values you determined for the mass and radius as a function of central density.
3. Plot of the density vs. radius for one of the cases in your table.
4. Any additional calculations you did to assess the accuracy of your code. (Hint: did you choose an appropriate value for the step size in radius? How could you check this?)