Solutions to the Homework #9  
(Due April 19)

1. Griffiths 8.15

\[ \int_0^1 F_2 \text{neutron} \, dx = 0.12 \quad \int_0^1 F_2 \text{proton} \, dx = 0.18 \]

Let's prove that \( \int_0^1 x d\sigma(x) \, dx = 2 \int_0^1 x d\pi(x) \, dx \).

\[ F_2 \text{proton} = x \left\{ 4 \left( \frac{2}{3} \right)^2 u \text{proton}(x) + \left( \frac{1}{3} \right)^2 d \text{proton}(x) \right\} = x \left\{ \frac{4}{3} u^p(x) + \frac{1}{3} d^p(x) \right\} \]

\[ F_2 \text{neutron} = x \left\{ 4 \left( \frac{2}{3} \right)^2 u \text{neutron}(x) + \left( \frac{1}{3} \right)^2 d \text{neutron}(x) \right\} = x \left\{ \frac{4}{3} u^n(x) + \frac{1}{3} d^n(x) \right\} \]

Due to \( \int_0^1 F_2 \text{proton} \, dx = \frac{3}{2} \int_0^1 F_2 \text{neutron} \, dx \), we have

\[ \int_0^1 x \left[ \frac{4}{3} u^p(x) + \frac{1}{3} d^p(x) \right] \, dx = \frac{3}{2} \int_0^1 x \left[ \frac{4}{3} u^n(x) + \frac{1}{3} d^n(x) \right] \, dx \]

\[ \int_0^1 x \left[ \frac{4}{3} u^n(x) + \frac{1}{3} d^n(x) - \frac{4}{3} u^p(x) - \frac{1}{3} d^p(x) \right] \, dx = 0 \]

Due to ud symmetry \( u^p(x) = d^n(x) \) and \( u^n(x) = d^p(x) \).

Finally, we have

\[ \int_0^1 x \left[ \frac{4}{18} u^p(x) - \frac{10}{18} d^p(x) \right] \, dx = 0 \]

\[ \int_0^1 x \, d\sigma(x) \, dx = 2 \int_0^1 x \, d\pi(x) \, dx \]
2. Griffiths 8.17

\[ F_2(x) = x \left[ \left( \frac{1}{3} \right)^2 u(x) + \left( \frac{1}{3} \right)^2 d(x) \right] \]

The data on the figures are not compatible with the equation. They are inconsistent because the idea, that quarks inside the proton are free, is wrong.

In the area \( x < 0.2 \), in figure (+x) \( d(x) \) and \( d(x) \) are positive, but in figure (-x) the slope \( \frac{dF_2}{dx} < 0 \).

3. Griffiths 10.2

Let's calculate the lifetime of the \( \tau \) lepton.

\[ \tau^- \rightarrow \mu^- \bar{\nu}_\mu \bar{\nu}_e \]
\[ \tau^- \rightarrow e^- \bar{\nu}_e \bar{\nu}_e \]

\[ \tau^+ = \left( \frac{m_\tau g_\mu}{M_W} \right)^2 \frac{m_e c^2}{12 \pi (87)^{3/2}} \]  

The lifetime \( \tau_{\tau^+} = \frac{1}{\lambda} = \left( \frac{M_W}{m_\tau g_\mu} \right)^4 \frac{12 \pi (87)^{3/2}}{m_e c^2} \)

Substituting all values \( g_\mu = 0.66 \), \( M_W = 80.4 \text{ GeV} \), \( m_\tau = 1.784 \text{ GeV} \)

Finally \( \tau_{\tau^+} = 1.528 \times 10^{-12} \) s

Taking into account that we have two decay modes we double the answer:

\( \tau_{\tau^-} = 3 \times 10^{-12} \) s

From Particle Data Group list \( \tau_{\tau^-} = 2.9 \times 10^{-13} \) s, which is very close to our calculation.
\[ M = \frac{G_F}{\sqrt{2}} c, \bar{f}, \left( p^\mu + q^\mu \right) \left[ \bar{u}(p) \gamma^\mu (1 - \gamma_5) \sigma^\mu v(q) \right] = \]
\[ = \frac{G_F}{\sqrt{2}} c, \bar{f}, \left[ \bar{u}(p) \gamma^\mu (1 - \gamma_5) \sigma^\mu v(q) + \bar{u}(p) (1 + \gamma_5) \not{v}(q) \right] \]

Using Dirac equation:
\[ (\gamma^\mu + m_\pi^\mu) v(x) = 0 \Rightarrow \not{v}(x) = -m_\pi^\mu = 0 \]
\[ \bar{u}(p) (\gamma^\mu - m_\pi^\mu) = 0 \Rightarrow \bar{u}(p) \not{v} = \bar{u}(p) m_\pi \]

we have:
\[ M = \frac{G_F}{\sqrt{2}} c, \bar{f}, m_\pi \bar{u}(p) (1 - \gamma_5) \sigma^\mu v(q) \]
\[ \left| M \right|^2 = \frac{G_F^2}{\sqrt{2}} c, f, \sum_{\text{spins}} m_\pi^2 \left[ \bar{u}(p) (1 - \gamma_5) v(q) \right] \left[ \bar{v}(q) (1 + \gamma_5) u(p) \right] = \]
\[ = 4 G_F^2 c, \bar{f}, m_\pi^2 (p \cdot q) \]

Due to \( E_0 = 1 \bar{p}^\mu p_\mu \Rightarrow E_0 + E_\pi = m_\pi = \left( m_\pi^2 + E_\pi^2 \right)^{1/2} \Rightarrow E_\pi = m_\pi - E_0 \]
\[ \Rightarrow E_0 = \frac{m_\pi^2 - m_\pi^2}{2m_\pi} \]

\[ \text{Hence } p \cdot q = E_\pi (E_0 - E_0 \cos \theta) \Rightarrow p \cdot q = E_\pi, \cos \theta = \frac{E_\pi}{E_0} \]

\[ \text{Also } \sin \theta = \frac{E_\pi}{E_0} \]

\[ \text{Combining all together we have } f_\pi^2 = \frac{4G_F^2 c, \bar{f}, m_\pi^2}{m_\pi^2 c, \bar{f}, \cos^2 \theta} \left( \frac{1}{2m_\pi} \right)^2 \]
(3) Griffiths, 10.12

\[
\frac{\Gamma (K^- \rightarrow e^- \bar{v}_e)}{\Gamma (K^- \rightarrow \mu^- \bar{\nu}_\mu)} = \frac{m_e^2 (m_e^2 - m_\mu^2)}{m_\mu^2 (m_e^2 - m_\mu^2)^2}
\]

Substituting \( m_e = 0.511 \text{ MeV}/c^2 \)
\( m_\mu = 105.66 \text{ MeV}/c^2 \)
\( m_e = 493.62 \text{ MeV}/c^2 \)

we have \( \frac{\Gamma (K^- \rightarrow e^- \bar{v}_e)}{\Gamma (K^- \rightarrow \mu^- \bar{\nu}_\mu)} = 2.57 \times 10^{-5} \)

Let's estimate \( f_K \) using information that \( \tau_K = 1.2 \times 10^{-8} \text{ s} \)

\[
\Gamma (K^- \rightarrow \mu^- \bar{\nu}_\mu) = 0.64 \frac{\tau_K}{\text{tot}} = 0.64 \frac{1.2 \times 10^{-8}}{1.2 \times 10^{-8} \text{ s}} = 6.33 \times 10^{-5} \text{ s}^{-1}
\]

\[
\Gamma (K^- \rightarrow \mu^- \bar{\nu}_\mu) = \frac{f_K^2}{4 \pi m_K^3} \left( \frac{g \mu}{4 \pi \alpha} \right)^2 m_\mu^2 (m_e^2 - m_\mu^2)^2
\]

\[
f_K = \sqrt{\frac{\Gamma (K^- \rightarrow \mu^- \bar{\nu}_\mu) 4 \pi m_K^3 (4 \mu)^2}{g \mu^2 m_\mu^2 (m_e^2 - m_\mu^2)^2}} = 38.22 \text{ MeV}
\]

From Particle Data Group
\( f_K \approx 160 \text{ keV} \).