Prove

$$\sum_{m=0}^{n} \binom{n}{m} = 2^n \tag{1}$$

using the well known recursive property of Pascal's Triangle (not proven here)

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1} \tag{2}$$

where

$$\binom{n}{m} \stackrel{\text{def}}{=} \frac{n!}{m!(n-m)!} \tag{3}$$

To simplify this proof, we will substitute $n \to n+1$ in equation (1), i.e. we will prove

$$\sum_{m=0}^{n+1} \binom{n+1}{m} = 2^{n+1} \tag{4}$$

and then generalize this result to equation (1).

First, we note that for $n+1=0$ and $n+1=1$, equation (4) is easily proven since there are one and two terms, respectively, in the sum. For $n+1>1$, we find

$$\begin{aligned}\sum_{m=0}^{n+1} \binom{n+1}{m} &= \frac{(n+1)!}{0!(n+1)!} + \sum_{m=1}^{n} \binom{n+1}{m} + \frac{(n+1)!}{(n+1)!0!} \\ &= 1 + \sum_{m=1}^{n} \binom{n+1}{m} + 1 \\ &= 2 + \sum_{m=1}^{n} \binom{n+1}{m}\end{aligned} \tag{5}$$

Let's look at the second term and use equation (2).

$$\begin{aligned}\sum_{m=1}^{n} \binom{n+1}{m} &= \sum_{m=1}^{n} \binom{n}{m} + \sum_{m=1}^{n} \binom{n}{m-1} \\ &= \sum_{m=1}^{n} \binom{n}{m} + \sum_{m=0}^{n-1} \binom{n}{m} \\ &= \sum_{m=1}^{n-1} \binom{n}{m} + \frac{n!}{n!0!} + \frac{n!}{0!n!} \sum_{m=1}^{n-1} \binom{n}{m} \\ &= 2 + 2\sum_{m=1}^{n-1} \binom{n}{m}\end{aligned} \tag{6}$$

Notice that the sum in the second term of our result of equations (6) is similar in structure to the second term of our result of equations (5). Each iteration of applying equation (2) nests another set of equations similar to the result of equations (6).

We end up with the result

$$\begin{aligned}\sum_{m=0}^{n+1} \binom{n+1}{m} &= 2 + 2\left(2 + 2\left(2 + \ldots 2\left(2 + 2\sum_{m=1}^{1} \binom{2}{m}\right)\ldots\right)\right) \\ &= 2 + 2\,(2 + 2\,(2 + \ldots 2\,(2 + 2\,(2))\ldots)) \\ &= 2(2^n - 1) \\ &= 2^{n+1} - 2\end{aligned} \tag{7}$$

Combining the result of equations (7) with the result of equations (6) we have

$$\begin{aligned} \sum_{m=0}^{n+1} \binom{n+1}{m} &= 2 + 2^{n+1} - 2 \\ &= 2^{n+1} \end{aligned} \tag{8}$$

which is equation (4). We can generalize equation (4) to get equation (1) by substituting $n \to n+1$, which is the end of the proof.