

THE ONSET OF CHAOS

PURPOSE

To study the onset of chaos in a nonlinear RLC circuit and to explore the approach to chaos demonstrated by the Logistic Equation.

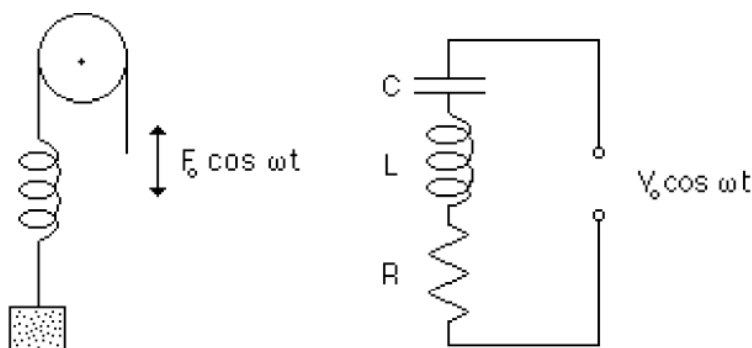
REFERENCES

"Period Doubling and Chaotic Behavior in a Driven Anharmonic Oscillator", Paul Lindsay, [*Physical Review Letters*, 47 1349 \(1981\)](#).

"Mathematical methods for Scientists and Engineers", Peter Kahn, Chapter 16. Q158.5.K34 1990.

BACKGROUND

A few weeks ago you did an experiment on the forced harmonic motion of a resonant spring-mass system, which demonstrated several important features: (1) the mass undergoes a transient initial motion determined by the initial conditions (such as initial displacement and velocity), (2) the transient motion damps out leaving a steady state motion independent of the initial conditions, (3) the steady state motion is a linear function of the amplitude of the driving force (i.e. if the force is doubled, the amplitude of oscillation doubles). This last feature is a characteristic of a linear system. You also may have done or studied an analogous experiment on a resonant series RLC circuit, where L would play the role of the mass, $1/C$ that of the spring constant, and the charge Q stored on the capacitor the role of the amplitude of oscillation, which showed exactly the same behavior.



In this experiment you will study a series resonant RLC circuit, but with one crucial change. In place of using a capacitor whose capacitance is constant, you will use one whose capacitance depends on the magnitude of the voltage across it. This capacitor voltage will then no longer be a linear function of the driving voltage and we have a nonlinear or anharmonic system.

You will monitor the voltage V_R across the resistance in the circuit shown above as you increase V_O . At first you will observe a voltage oscillating at the frequency ω with each successive peak (crest) of the oscillation having the same magnitude. But when V_O becomes large enough (larger than a certain threshold value), the peaks will alternate in amplitude, i.e., V_R will no longer have period $T_1 = 2\pi/\omega$, but rather twice the period, $T_2 = 4\pi/\omega = 2T_1$. There has been a period doubling of the response of the circuit. As V_O is further increased (larger than another threshold value), the pattern of peaks again changes so that the same amplitude only occurs every fourth cycle, i.e. there has been another period doubling and the period is now $T_3 = 8\pi/\omega = 4T_1$. As V_O is still further increased there is a succession of period doublings which require smaller and smaller increments in V_O . You will rapidly find that no matter how good your apparatus is, you cannot control V_O accurately enough to discern successive period doublings and the voltage across the resistor appears to vary randomly with time. But we know from what we've seen for smaller V_O that the response of the circuit varies in a perfectly well determined manner. This deterministic yet apparently random behavior is called **chaos** to distinguish it from truly random behavior.

It has been only since the advent of high speed computers that scientists have realized the extent to which nonlinear systems may exhibit chaotic behavior and that the approach to chaos through period doubling is a universal behavior that does not depend on the details of the system studied. In this experiment we will first observe the chaotic behavior of the nonlinear RLC circuit, and then use MatLab or Origin (or simply Excel) to study the approach to chaos for the Logistic Equation, an extremely simple nonlinear difference equation that exhibits chaos.

The Logistic Equation,

$$x(n+1) = a \cdot x(n) \cdot [1 - x(n)], \quad (1)$$

has been used to model natural populations such as the yearly population swings of lemmings. If the population $x(n)$ during the n -th interval is known, then Eq. (1) gives the population $x(n+1)$ in the next interval (i.e. the next year, for lemmings). a is called the growth parameter and is essentially the difference between the birth and death rates. [Note that here a plays the role of V_O in nonlinear RLC circuit.] We will allow $x(n)$ to vary between 0 and 1, which limits a to being between 0 and 4.

Just as for the oscillating mass or the RLC circuit, the Logistic Equation shows an initial behavior which depends on the starting condition $x(0)$ and damps out leaving a steady state response, i.e. a constant value for $x(n)$ for large n , when the value of a is not too large (see below for detail discussion). It is important to notice the difference between Logistic Equation and the RLC circuit, i.e. there is no driving term so that the "normal" steady state behavior will be a constant value rather than the steady state oscillation seen for the RLC circuit. Despite this difference the approach to chaos will be essentially the same as that of the RLC circuit.

The behavior of the steady state response of the Logistic Equation depends strongly on a as follows:

$0 \leq a < 1$: The growth parameter is too small to sustain any population. Regardless of $x(0)$ the population crashes to zero after a few intervals. $x(n) = 0$ is called a fixed point of the system

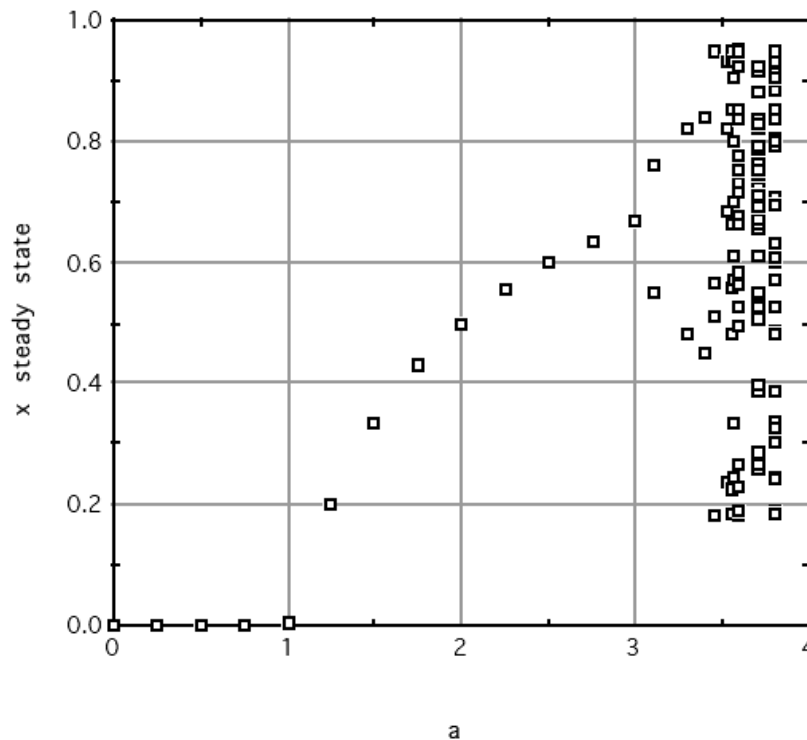
since $x(n+1) = x(n)$. It is a stable fixed point because after any small disruption the population returns to zero. Stable fixed points are also called attractive fixed points or attractors.

$1 \leq a < 3$: For $a = 1$ the fixed point at zero becomes unstable and is called a repellent fixed point or a repeller. At the same time a new attractor appears, $x(n) = 1 - 1/a$. The growth parameter is now large enough to sustain a nonzero steady state population. The transient solution monotonically approaches $x(n) = 1 - 1/a$ for a close to 1, but then for larger a it oscillates about the attractor before eventually being damped out.

$3 \leq a < 3.45$: At $a = 3$ the attractor $x(n) = 1 - 1/a$ become unstable and the steady state solution oscillates about $x(n) = 1 - 1/a$ with alternate $x(n)$ having the same value. For $a < 3$, the interval between like values for $x(n)$ in the steady state solution was $\Delta n = 1$, while for $a \geq 3$ the interval is $\Delta n = 2$; the period has doubled. Do not confuse the appearance of this oscillation with the steady state behavior of the RLC circuit before frequency doubling has set in.

$3.45 < a < 3.54$: At $a \approx 3.45$ the period doubles again so that the interval between equal values of $x(n)$ is now $\Delta n = 4$.

$3.54 < a < 3.6$: The steady state solution goes through a series of period doublings as shown in the graph on the next page until at $a \approx 3.6$ chaos sets in. The graph on the next page show this period doubling and approach to chaos.



In your calculations you will see that in the chaotic region, the steady state is so sensitive to the value of a that it is impossible for us to predict the solution. The slightest error in our determination of a will completely throw off our predicted behavior. Calculations with the Logistic Equation will also show that the steady state solution is no longer independent of the starting condition and is in fact extremely sensitive to the exact value of $x(0)$. We must conclude then, that when chaos appears and we cannot theoretically predict the response of a system even though it is in principle completely deterministic. We also noted that in a linear system or in the nonchaotic region for the Logistic Equation, that the steady state solution is independent of the starting conditions.

The sequence of period doublings that you will observe in this experiment for both the RLC circuit and the Logistic Equation has a universal character that is independent of the particular system. Most of the details of this universal behavior are beyond the scope of this laboratory. However, one aspect is accessible; Feigenbaum (see references in Phys. Rev. Lett. listed above) has demonstrated that if a_1 is the value of a where the first period doubling sets in, a_2 the value for the second doubling etc., then for large n

$$a_\infty = a_n + S\delta^{-n}, \quad (2)$$

where a_∞ is the value of a where chaos sets in. S is a constant characteristic of the particular system, but $\delta = 4.6692\dots$ is a universal constant called the Feigenbaum number. Since δ is large, this series converges rapidly and Eq.(2) will allow us to estimate δ and predict a_∞ with only a little data. In this experiment you will measure the first few a_n and use this data to estimate δ from Eq.(2). To do this use data from three successive period doublings to eliminate S and a_∞ from Eq.(2) and obtain

$$\delta = \frac{a_n - a_{n-1}}{a_{n+1} - a_n}. \quad (3)$$

If n is large enough, the value of δ calculated this way should equal Feigenbaum's theoretical value. To get a_∞ from your data, solve Eq.(2) for S ,

$$S = \frac{(a_n - a_{n-1})\delta^n}{\delta - 1}, \quad (4)$$

and then use Eq.(2) to get a_∞ .

EQUIPMENT

The RLC circuit is pre-wired for you. For the nonlinear capacitor we use a silicon diode. The diode consists of a layer of p-type silicon in contact with a layer of n-type silicon. When the diode is reverse biased the positive charges in the p-type silicon and the negative charges in the n-type silicon are moved away from the junction by the electric field produced by the applied voltage. When the junction is forward biased they are moved toward the junction. Thus the distance of separation of the charge in the diode and hence, its capacitance depends on the

voltage. It turns out that this dependence is very nonlinear. The voltage applied across the diode will not be large enough to turn the diode on and make it stop acting as a capacitor. The equipment needed is:

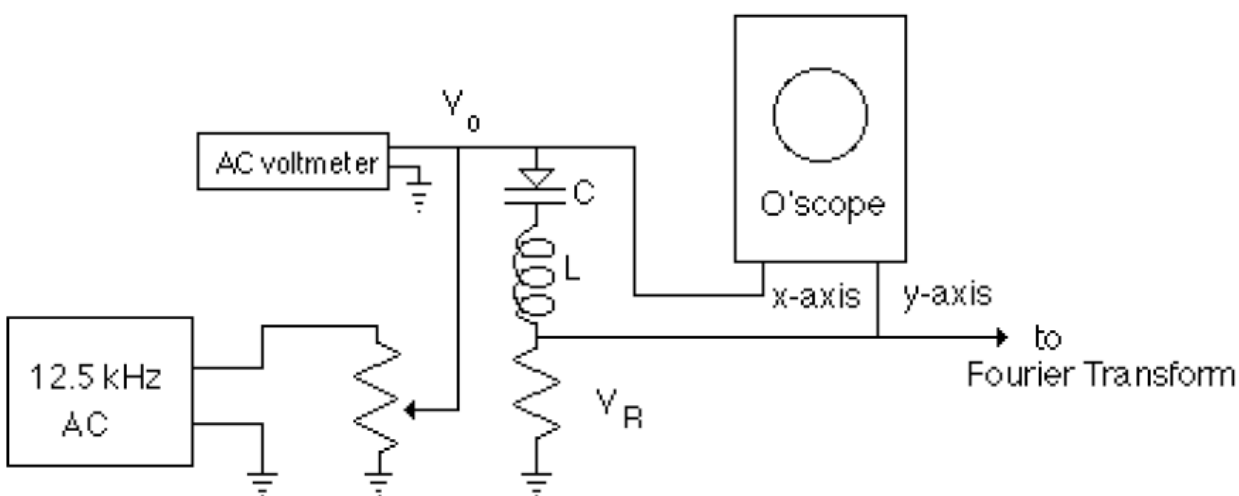
RLC circuit

Oscilloscope (TDS 2012C)

Function Generator (Agilent 33210A)

SR760 FFT spectrum analyzer

4 digit AC voltmeter



PROCEDURE

Part 1: Nonlinear RLC circuit

1. Set up the nonlinear RLC circuit as shown below:
2. Set the function generator to output a sine wave with frequency of ~10-12 kHz and maximum amplitude (about 10 V). When you use your oscilloscope, you will use two different Display modes ("YT" and "XY" modes) to display V_R in a useful manner. You can either look at V_R versus time (the "YT" mode) using the time sweep for the x-axis, or you can look at V_R versus V_0 ("XY" mode), which is known as a Lissajous figure. You will find the Lissajous figure most useful for detecting frequency doubling. Practice using both modes of operation.
3. Adjust V_R to a small value by turning the knob on the right (i.e., NOT the one labeled Drive Voltage) counter-clockwise until it stops then turn back about one-half turn. Adjust the drive level (V_0) until the voltage on the voltmeter reads about 0.1 V. Look at V_R on the oscilloscope. Adjust the frequency until the amplitude of V_R is a maximum.
4. Look at the spectrum on the FFT. Use log scale and dBV as unit. Adjust the frequency span to 12.5 kHz. You should see a peak around 9-12 kHz. Any other peaks should be smaller than the

main peak by at least 40 db. You may see two peaks at about half the driving frequency. If so, adjust slightly the driving frequency until the two peaks merge.

5. Set the oscilloscope to look at the Lissajous figure. Adjust the Volts/div scales to give an ellipse about two divisions in diameter. Starting with small V_O , slowly increase its value and look for evidence for period doubling using the Lissajous figure.

Q: What happens to the FFT spectrum when this happens?

Observe the amplitude and frequencies of peaks in the FFT spectrum. As you increase V_O still further, observe additional doubling and the onset of chaos. You may notice that once you have reached the chaotic region, there are still values of V_O for which you get simple deterministic behavior (period tripling and quintupling). Import an averaged 16 peak spectrum into Origin and present it to your report.

6. Using the AC voltmeter to measure V_O and the oscilloscope to measure the positive peak value(s) of V_R , prepare a graph of V_R versus V_O , equivalent to the earlier graph for period doubling for the Logistic Equation. Record a few points for each stable interval. Carefully determine the values of V_O where period doubling occurs (easier with the spectrum analyzer) and use Eqs. (2)-(4) to estimate a_∞ , S , and δ . [Note that here V_O plays the role of the growth parameter a . So a_∞ means $V_O(\infty)$ where the chaos sets in.] Once the frequency is close to a doubling point, average the spectrum over 50 samples to get a better sensitivity for better determining the actual critical point. For the 3rd or higher points you should try setting the frequency span to half to better isolate the point.

Part 2: the Logistic Equation

1. Set up a MatLab m-file (or Origin worksheet or Excel spread sheet) to calculate $x(n)$ for **various values** of $x(0)$ and a . In order to examine the transient response of the equation, graph $x(n)$ for the first 100 values of n , and to examine the steady state response, graph $x(n)$ for about 200 cycles (or more) when a is larger. (To get the steady state solution near $a = 3$, you may have to go up to $n \approx 1000$ or more to avoid the transient.) Place several plots on the same page in your lab report in order to see the effect of changing a or $x(0)$.

2. Examine the behavior of the solutions for $a < 3$ for various values of $x(0)$ in order to verify that the equation behaves as described in the BACKGROUND section. In particular verify that the steady state solution is **independent** of the initial condition.

3. Examine the behavior of the solutions for the four cases $a = 2.99$ and 3.01 with $x(0) = 0.667$ and 0.5 , where period doubling first sets in. Q: Does the steady state value depend on the initial value for any of these cases?

4. Determine to **four** decimal places (3.xxxx) for the values of a where period doubling sets for at least 4 period doublings. To do this, examine the steady state solution values of $x(n)$ for n , $n+1$, $n+2$, ... to see how many cycles you have to go through before the pattern repeats. Then vary a until you find the value where the period doubles. Use a value of $n > 1000$ and increase

the accuracy of $x(n)$ as much as necessary to see a change in pattern. Use Eqs. (2)-(4) to calculate a_∞ , S , and δ .

5. For a in the chaotic regime, $a > 3.6$, examine the steady state solution to see the dependence on extremely small changes in a and in $x(n)$. The effect is easiest to see on a small graph if you use values near 3.9. For example, use a starting point of 0.60 and record the value of $x(n)$ at the 200th iteration. Then change the starting point to 0.6001. Q: How much does the value of $x(n)$ 200th iteration change?