PRECISE RELATIONS BETWEEN THE SPECTRA OF VECTOR AND AXIAL-VECTORS MESONS

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Two sum rules are derived, relating moments of the spectral functions of the vector and axial-vector currents. If it is assumed that the ρ and A1 mesons dominate these moments, then their masses must be in the ratio $m_{A1}/m_{\rho} = \sqrt{2}$, in very good agreement with experiment.

If chiral SU(2)⊗SU(2) were an exact symmetry of the ordinary sort, we should expect the ρ meson to be accompanied with an I = 1 axial-vector meson of the same mass. This is certainly not the case; the best candidate for the role of chiral partner of the ρ is the A1, which has $m_{A1}^2 = 2m_{\rho}^2$. However, the recent successes of current algebra show that nature does obey some sort of chiral symmetry, manifested in the conservation or partial conservation of currents, and in their commutation relations. The question thus arises: What relations are imposed by current algebra upon the spectra of the 1+ and 1− mesons?

Our answer is contained in the following theorem: Assume that the vector and axial-vector currents obey the usual commutation relations, with Schwinger terms which are either c numbers or, if operators, contain no Δ = 1 terms. Neglect the pion mass altogether, so that the axial vector as well as the vector currents are conserved. Then

$$\int_0^\infty [\rho_v(\mu^2) - \rho_{A1}(\mu^2)] d\mu^2 = F_\pi^2,$$

(1)

where $F_\pi$ is the usual pion-decay amplitude, and $\rho_v, \rho_A(\mu^2)$ are the spectral functions of the vector and axial-vector currents, defined by the formulas,

$$\langle \nu_a(\mu^2) V_\nu(0) \rangle = (2\pi)^{-3} \delta_{ab} \int d^4p \rho(p) e^{i p \cdot x} \rho_v(-p^2) \rho_{A1}(-p^2) [\epsilon^{\mu\nu} - \rho^{\mu\nu} / p^2],$$

(2)

$$\langle A_a(\mu^2) A_\nu(0) \rangle = (2\pi)^{-3} \delta_{ab} \int d^4p \rho(p) e^{i p \cdot x} \rho_v(-p^2) \rho_{A1}(-p^2) [\epsilon^{\mu\nu} - \rho^{\mu\nu} / p^2] + F_\pi^2 \rho^{\mu\nu} p^{\mu} p^{\nu}. \rho_v^{\nu}.$$ (3)

If we further assume a very weak form of vector- and axial-vector-meson dominance, i.e., that matrix elements of the currents act at high momenta as if the currents were free 1+ fields, then we also have

$$\int_0^\infty [\rho_v(\mu^2) - \rho_{A1}(\mu^2)] d\mu^2 = 0.$$ (4)

Before proving these theorems, let us note some of their implications. The spectral functions $\rho_v, \rho_A(\mu^2)$ are measurable, in principle, from the cross sections for hadron production in electron-neutrino collisions. For the present, we can estimate $\rho_v(\mu^2)$ by using the hypothesis of $\rho$ dominance:

$$\rho_v(\mu^2) \approx g_{\rho}^2 \delta(\mu^2 - m_{\rho}^2).$$ (5)

Eqs. (1) and (4) now read

$$\int_0^\infty \rho_{A1}(\mu^2) d\mu^2 \approx g_{\rho}^2 m_{\rho}^{-2} F_\pi^2,$$

(6)

Hence, if $\rho_{A1}(\mu^2)$ is sharply peaked about a point $\mu = m_{A1}$, we must have

$$m_{A1} / m_{\rho} \approx [1 - F_\pi^2 m_{\rho}^2 / g_{\rho}^2]^{-1/2}. \quad (8)$$

Using $\rho$ dominance and either current algebra or the observed $\rho$ width, we have $g_{\rho}^2 \approx 2F_\pi^2 m_{\rho}^2$, so Eq. (8) gives

$$m_{A1} / m_{\rho} \approx \sqrt{2},$$

(9)

in extraordinary agreement with the observed masses of the ρ and A1, for which $m_{A1} / m_{\rho} \approx 1.41 \pm 0.01$.

Now to the proof of Eqs. (1) and (4). Define

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a three-point function

\[ -i \epsilon^{abc} M^{\mu \nu \lambda} (q, p) = \int d^4 x d^4 y \langle T [A^\mu_a (x), A^\nu_b (y), V^\lambda_c (0)] \rangle_0 \exp \left[ -i q \cdot x - i p \cdot y \right]. \]  

Our assumptions lead immediately to the following Ward identities:

\[ \frac{1}{2} q \cdot M^{\mu \nu \lambda} (q, p) = \Delta V^{\mu \nu} (q + p) \Delta_A^{\lambda \nu} (p), \]  

\[ \frac{1}{2} (q + p) \cdot M^{\mu \nu \lambda} (q, p) = \Delta_A^{\mu \nu} (q) \Delta_A^{\nu \lambda} (p), \]

where \( \Delta_V \) and \( \Delta_A \) are the propagators of the vector and axial-vector currents. Multiply (11) with \( (q + p) \lambda \) and (12) with \( q \mu \) and subtract; this gives

\[ (q + p) \cdot \Delta_V^{\mu \lambda} (q + p) = q \cdot \Delta_A^{\mu \lambda} (q) + p \cdot \Delta_A^{\nu \lambda} (p). \]  

Equation (13) holds for all values of \( q \mu \) and \( p \nu \), so each term must be the same linear function of its argument, i.e.,

\[ K^{\lambda \nu} (q) = K^{\mu \lambda} (q) = C^{\mu \nu}, \]

with \( C \) a constant. Writing this in coordinate space and using current conservation, Eq. (14) becomes

\[ \delta(x) \langle [V^0_a (x), V^\nu_b (0)] \rangle_0 = \delta(x) \langle [A^0_a (x), A^\nu_b (0)] \rangle_0 = -i \delta_{ab} C^{\nu \lambda} \delta^4 (x), \]

so our theorem states the equality of the vector and axial-vector Schwinger terms! The vacuum expectation values in (15) can be evaluated from (2) and (3), with the result that they vanish for \( \nu = 0 \), while for \( \nu = 1, 2, 3 \) they are

\[ \delta(x) \langle [V^0_a (x), V^\nu_b (0)] \rangle_0 = -i \delta^\nu \delta^4 (x) \rho^{(2)} (\mu^2) \frac{d^2}{d \mu^2}, \]  

\[ \delta(x) \langle [A^0_a (x), A^\nu_b (0)] \rangle_0 = -i \delta^\nu \delta^4 (x) \left[ F^{(2)} + \frac{d^2}{d \mu^2} \rho^{(2)} (\mu^2) \right]. \]

Equation (1) now follows from (15)-(17).

In order to prove Eq. (4) we return to Eq. (11) and now set \( q^\mu = 0 \). The left-hand side has a one-pion pole, which is all that survives at \( q^\mu = 0 \), so Eq. (11) now reads

\[ \frac{1}{2} F \int d^4 x \langle \pi \cdot T [A^\nu_b (y), V^\lambda_c (0)] \rangle_0 \exp \left[ -i p \cdot y \right] = \Delta_V^{\mu \nu} (p) - \Delta_A^{\mu \nu} (p), \]

where \( \langle \pi_a \rangle \) is a covariantly normalized state representing a pion of zero energy and isospin index \( a \). Next let \( p^2 - \infty \). Our assumption\(^a\) that the currents behave at infinity like free fields tells us that the coefficient of \( g^{\mu \nu} \) on the left-hand side behaves like \( (p^2)^{-2} \), while the coefficient of \( g^{\mu \lambda} \) on the right-hand side approaches

\[ \frac{1}{p^2} \int_0^\infty \left[ \rho^{(2)} (\mu^2) - \rho^{(2)} (\mu^2) \right] d \mu^2, \]

so Eq. (4) is necessary for the consistency of Eq. (18). Precisely the same reasoning applies if we approximate \( \Delta_V \) and \( \Delta_A \) by sums over

meson poles, and approximate the left-hand side of Eq. (18) by a double sum over these poles.

Our new sum rules (1) and (4) are distinguished from those of the Adler-Weisberger type, in that they do not seem to have anything to do with low-energy theorems, but deal instead with high-energy behavior and, surprisingly, with the Schwinger terms. Another distinguishing feature of practical importance is that the integrals in (1) and (4) receive contributions only from states of fixed spin and isospin; this...
is presumably why our assumption of $\rho$ and A1 saturation works so well.

The methods used here can obviously be applied to the currents of larger groups, like SU(3) $\otimes$ SU(3), and to the higher $n$-point functions of the currents.

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1M. Gell-Mann, Phys. Rev. 131, 63 (1964).


3H. Rosenfeld et al., University of California Radiation Laboratory Report No. UCRL-8030 (unpublished). The fact that a $\rho$ enhancement is observed at $s = 2m_\rho^2$ does not necessarily force us to the conclusion that this is a single pure resonant state; all that is needed to derive Eq. (9) is that there is some particularly strong interaction in $1^+ J=1$ states near this energy, which dominates the axial-vector spectral function. In this form the assumption of A1 dominance is supported by data from neutrino experiments, since the best fit to the axial-vector form factor uses an effective $1^+$ meson mass significantly larger than $m_\rho$.

4S. Adler, private communication. There is some reason to be cautious in calling the A1 a pure resonance, because using current algebra to calculate the $\pi V A$ vertex at zero pion energy, and using $\rho$ and A1 dominance of the currents, yields a $\pi^0\rho\pi^{-}A$ coupling constant so large that the A1 width would be about 800 MeV, instead of the observed 125 MeV; D. Geffen, to be published; B. Renner, Phys. Letters 21, 453 (1966); H. Schnitzer, unpublished. [This result can also be immediately obtained from Eq. (18).] One way out of this difficulty is to suppose that the A1 is not a simple resonance, and that a Breit-Wigner representation of the A1 requires many levels. A more attractive possibility is that the $\pi^0\rho\pi^{-}A$ coupling constant that actually governs the rate of A1 decay, where the $\pi$ is not soft, is much less than that calculated for zero-energy pions.

5This assumption will hold approximately to the extent that the currents are dominated by a finite number of meson poles in the sense of M. Gell-Mann and F. Zachariasen, Phys. Rev. 124, 953 (1961). Their work also suggests that our assumption about the high-momentum limit, and hence Eq. (4), may become exact, if $J=1$ fields are coupled to the currents (as advocated by J. J. Sakurai, Ann. Phys. (N.Y.) 11, 1 (1960)) and we let their bare mass go to infinity. If the $\rho$ and A1 dominate $\rho_{\pi^{-}}$ and $\rho_{\pi^{-}}\pi^{-}$, then Eq. (4) says that the coupling of the $\rho$ and A1 to the $V$ and $A$ currents are the same, a result derived independently by D. Geffen (to be published).

6This formula was originally obtained by the author in the course of developing a Lagrangian, involving $1^+$ and $1^-$ Yang-Mills fields, which reproduces the soft-pion results of current algebra. Note that if there were no massless pion, or if it decoupled from other particles so that $F_\pi=0$, then we would recover the prediction of naive chirality, i.e., $m_A=m_\rho$. This is one more instance of the Goldstone theorem.


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