Advanced General Physics I

Physics 323

Notes on Linear Algbra

Fall 2019

In general, the best way to solve normal mode problems is to use linear algebra. For those of you interested, here is how to do that.

Given a problem in which the equations of motion for the coordinated are coupled, we want to find the normal coordinates and the frequency of the normal modes. First, let treat the case in which there are two degrees of freedom. The equations of motion for the coordinates will be of the form:

$$\ddot{x}_1 = a_{11}x_1 + a_{12}x_2$$
$$\ddot{x}_2 = a_{21}x_1 + a_{22}x_2$$

For normal mode motion, the two coordinate x_1 and x_2 will each under go simple harmonics motion with angular frequency ω, the frequency of the normal mode. Then:

$$x_1 = A_1 e^{i\omega t} \quad \Rightarrow \quad \ddot{x}_1 = -\omega^2 A_1 e^{i\omega t} = -\omega^2 x_1$$
$$x_2 = A_2 e^{i\omega t} \quad \Rightarrow \quad \ddot{x}_2 = -\omega^2 A_2 e^{i\omega t} = -\omega^2 x_2$$

We then have:

$$-\omega^2 x_1 = a_{11}x_1 + a_{12}x_2$$
$$-\omega^2 x_2 = a_{21}x_1 + a_{22}x_2$$

or rearranging:

$$(a_{11} + \omega^2)\, x_1 + a_{12}x_2 = 0$$
$$a_{21}x_1 + (a_{22} + \omega^2)\, x_2 = 0$$

$$(a_{11} + \omega^2)\, A_1 e^{i\omega t} + a_{12}A_2 e^{i\omega t} = 0$$
$$a_{21}A_1 e^{i\omega t} + (a_{22} + \omega^2)\, A_2 e^{i\omega t} = 0$$

$$(a_{11} + \omega^2)\, A_1 + a_{12}\, A_2 = 0$$
$$a_{21}\, A_1 + (a_{22} + \omega^2)\, A_2 = 0$$

We can write the last pair of equations in matrix form:

$$\begin{pmatrix} a_{11} + \omega^2 & a_{12} \\ a_{21} & a_{22} + \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This has a solution only if the matrix is non-invertible, that is, it does not have an inverse. We can see that since as follows. Let:

$$M = \begin{pmatrix} a_{11} + \omega^2 & a_{12} \\ a_{21} & a_{22} + \omega^2 \end{pmatrix}$$

If the matrix M were to have an inverse M^{-1}, then $M^{-1}M = I$ where I is the identity matrix.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We would then have:

$$\begin{pmatrix} a_{11} + \omega^2 & a_{12} \\ a_{21} & a_{22} + \omega^2 \end{pmatrix}^{-1} \begin{pmatrix} a_{11} + \omega^2 & a_{12} \\ a_{21} & a_{22} + \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

But this would meant that

$$\begin{pmatrix} a_{11} + \omega^2 & a_{12} \\ a_{21} & a_{22} + \omega^2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

which is not possible. So M does not have an inverse. But, a matrix is non-invertible if and only if it's determinant is zero. So we have that:

$$\begin{vmatrix} a_{11} + \omega^2 & a_{12} \\ a_{21} & a_{22} + \omega^2 \end{vmatrix} = 0$$

This gives a quadratic equation for ω^2:

$$(a_{11} + \omega^2)(a_{22} + \omega^2) - a_{11}a_{12} = 0$$

We can solve this to obtain two solution for ω^2. In each case we keep only the positive solutions since a frequency is always positive. This gives us the frequencies of the two normal modes.

Once we have the frequency of a normal mode, we can find the corresponding normal coordinate by solving either

$$(a_{11} + \omega^2)\, A_1 + a_{12}\, A_2 = 0 \qquad \text{or} \qquad a_{21}\, A_1 + (a_{22} + \omega^2)\, A_2 = 0$$

for A_2 in terms of A_1.