This is a closed book/notes exam. A one-sided 8.5x11 sheet with only formulae is allowed. Please submit the sheet with your bluebook, but you may keep the exam. No calculators/phones. Tackle any (simple) calculations on your own. Total credit 30 points.

1. (2 points each, 4 total.) Please be concise.
   (i) Which nuclear process is the source of helium on the sun? (Nuclear fusion of hydrogen.)
   Which nuclear process is the source of helium on the earth? (alpha decay of heavy elements.)
   (ii) How is resonant absorption achieved in Mossbauer effect, i.e., how is the energy lost to recoil made negligible? (When a nucleus frozen in a crystal lattice emits a (relatively low energy) photon, the photon recoils against the massive lattice.)

2. (1 point each, 10 total). True or false?
   i) Both SU(2) and SO(3) symmetries lead to angular momentum conservation. TRUE, SU(2) for spin and SO(3) for orbital.
   ii) The characteristic length of the nuclear force is one \( fm \). TRUE.
   iii) It is hazardous to collect large quantities of deuterium oxide because the deuterons in this heavy water will spontaneously fuse into helium (which is a very stable nucleus). FALSE, coulomb barrier.
   iv) Just as the group SU(2) describes spin 1/2, group SU(2S+1) describes spin S. FALSE, SU(2) works for all spins.
   v) The Pauli matrices are hermitian. TRUE.
   vi) The Pauli matrices are members of SU(2). FALSE, SU(2) members are unitary.
   vii) The Pauli matrices are generators of SU(2). TRUE.
   viii) There is direct experimental evidence for proton decay. FALSE.
   ix) There is direct experimental evidence for neutron decay. TRUE, a free neutron decays after about 15 minutes.
   x) The \( \beta \)-decay is a two-body process. FALSE, it is a 3-body decay.

3. (4 points.) Calculate the energy splittings for the S,P,D, and F levels due to a \(-2\alpha L \cdot \hat{S}\)
term ($\alpha > 0$) in the nuclear potential. Be sure to indicate which level is lower in energy, e.g., when you consider $P_{1/2}$ and $P_{3/2}$.

Soln: Squaring $\vec{J} = \vec{L} + \vec{S}$, we get $-\alpha < 2\vec{L} \cdot \vec{S} >= -\alpha < \vec{L}^2 + \vec{S}^2 - \vec{J}^2 >= -\alpha \hbar^2[\ell(\ell + 1) + s(s + 1) - j(j + 1)]$. Nothing happens to the $S$ state since $\ell = 0$, but the splits for the other states, in units of $\alpha \hbar^2$, are $[(\ell + 1/2)(\ell + 1/2 + 1) - (\ell - 1/2)(\ell - 1/2 + 1)] = 2\ell + 1$. So the splits for $P$, $D$, and $F$ states ($\ell = 1, 2, \text{and} 3$), in units of $\alpha \hbar^2$, are 3, 5, and 7, respectively. For a given $\ell$, higher $j$ results in lower energy due to the negative sign of the coupling term.

4. (4 points.) A parent nucleus with a lifetime of 1.0 billion ($=10^9$) days decays to a daughter nucleus, which in turn decays with a lifetime of one day. Initially there $10^{23}$ parent nuclei in a sample, but no daughter nuclei. Approximately how many daughter nuclei are present after (a) 8.64 seconds (b) 1000 days? Notes: There are 86400 seconds in a day. Lifetime, not half-life. Differential equations not needed.

Soln: These are clearly two extreme cases. In the first case, the daughter nucleus production has just begun to accumulate and its decays can be ignored (8.64s is much less than 1 day) and in the second case, the production and decay are in equilibrium (1000 days is much more than one day). For both cases, the times involved are so small compared to the parent nucleus lifetime that its population stays unchanged. There are $10^{-9} \times 10^{23} = 10^{14}$ parent decays per day, so that many daughters are produced per day. 8.64s is $10^{-4}$th of a day, so there will be $10^{10}$ daughters after 8.64s. The daughter population will have reached an equilibrium number $N$ after 1000 days which means that $(10^{14}/\text{day})x(\delta t)$ daughters produced in a small time interval $\delta t$ days must equal the $(N)x(1/\text{day})x(\delta t)$ number of daughters that decay in that period. Equating the two, $N=10^{14}$.

5. (4 points.) An example of a neutron-induced fission reaction is $n + ^{235}_{92}U \rightarrow ^{93}_{37}Rb + ^{141}_{55}Cs + 2n$. Using the mass values listed below, estimate the energy released in this reaction in MeVs. The masses are in atomic mass unit (amu), which is 931.5 MeV/c$^2$. Since calculators are not allowed, this simple problem is really an exercise in handling significant figures and approximate calculations.

$m_n = 1.0086649, M(^{93}_{37}Rb) = 92.9220328, M(^{141}_{55}Cs) = 140.9200440, M(^{235}_{92}U) = 235.0439231$.

Soln: The mass of one neutron can be subtracted from the energy of the initial and final states so that the energy released is $(M(^{235}_{92}U) - M(^{93}_{37}Rb) - M(^{141}_{55}Cs) - m_n)c^2$.
\[
= (235.0439231 - 92.9220328 - 140.9200440 - 1.0086649) \text{amu} c^2
\]
\[
= 0.19318 \text{amu} c^2 \times 931.5 \text{MeV/(amu} c^2) = 180.0 \text{MeV}. \text{ The crux of the calculation is 5.04-2.92-2.92-1.01 = +0.04-(-0.08)-(-0.08)-0.01 = 0.19. Then multiply this 0.2 by 900.}
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6. (4 points.) Consider an “equation of motion” \( \nabla^2 \psi = -k^2 \psi \).

a) Does it obey the global U(1) gauge symmetry? (Consider \( \psi \rightarrow \psi \exp(i\alpha) \), where \( \alpha \) is a real number.)

b) Does it obey the local U(1) gauge symmetry? (This time make \( \alpha \) a real function.)

c) This equation also fails to include an interaction (force). Incorporate a divergence-free force \( \vec{F} \) \( (\nabla \cdot \vec{F} = 0) \) i.e., modify the equation of motion to make it obey local U(1) symmetry by including this interaction.

d) Verify that your modified equation obeys local U(1) gauge symmetry. It is ok to stop after showing that the phase term “goes through” the first derivative.

Soln: This problem is about the local U(1) symmetry origin of the electromagnetic interaction. Please look at your class notes or the text. Since the force is divergence-free, it can be written as the gradient of a vector potential \( \vec{A} \). Thus, the interaction is unchanged if \( \vec{A} \) is shifted by the gradient of some scalar function \( \alpha \).

a) The equation is symmetric under global U(1) because the multiplicative constant phase cancels from both sides.

b) It does not obey the local gauge symmetry because the derivatives of \( \alpha \) give an extra LHS term, so if \( \psi \) is a solution, it generally won’t be a solution when transformed.

c) We simply replace the \( \nabla \) by \( \nabla - i\vec{A} \) in the equation of motion.

d) The local gauge transformation now transforms both \( \psi \) (by a multiplicative phase factor) and \( \vec{A} \) (by addition of \( \nabla \alpha \)). Verify that the extra term obtained in differentiating the new \( \psi \) cancels the \( \nabla \alpha \) term and the equation becomes invariant under local U(1).