Physics 417(QM) HW7 Solutions

1 4.19

The potential energy is

\[ V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r} \]  

(1)

So the Bohr energies become

\[ E_n(Z) = -\frac{1}{n^2} \frac{m}{2\hbar^2} \left( \frac{Ze^2}{4\pi\varepsilon_0} \right)^2 \]  

(2)

The binding energy becomes

\[ E_1(Z) = -\frac{m}{2\hbar^2} \left( \frac{Ze^2}{4\pi\varepsilon_0} \right)^2 = -Z^2 \cdot 13.6eV \]  

(3)

Using 4.55 and 4.68, change \( e^2 \) into \( Ze^2 \),

\[ a(Z) = \frac{4\pi\varepsilon_0 \hbar^2}{mZe^2} = \frac{1}{Z} \cdot 0.529 \times 10^{-10}m \]  

(4)

The Rydberg constant becomes

\[ R(Z) = \frac{m}{4\pi\varepsilon_0 \hbar^3} \left( \frac{Ze^2}{4\pi\varepsilon_0} \right)^2 = Z^2 \cdot 1.097 \times 10^7 m^{-1} \]  

(5)

For \( Z = 2 \) and \( Z = 3 \) the Lyman series wavelengths become 1/4 and 1/9 times that for the hydrogen atom and land in the ultraviolet between 23-30 nm and 10-14 nm, respectively.

2 4.22

(a).

\[ L_i = \sum_{ijk} \varepsilon_{ijk} r_j p_k \]  

(6)
\[
[L_z, r_l] = \sum_{zjk} \varepsilon_{zjk} r_j [p_k, r_l] = -i\hbar \sum_{zjk} \varepsilon_{zjk} r_j \delta_{kl}
\]

\[
= \begin{cases} 
  iy & l = x \\
  -ix & l = y \\
  0 & l = z 
\end{cases}
\]  

(7)

\[
[L_z, p_l] = \sum_{zjk} \varepsilon_{zjk} [r_j, p_l] p_k = i\hbar \sum_{zjk} \varepsilon_{zjk} p_k \delta_{jl}
\]

\[
= \begin{cases} 
  i\hbar p_y & l = x \\
  -i\hbar p_x & l = y \\
  0 & l = z 
\end{cases}
\]  

(8)

\[
[L_z, L_x] = [L_z, yp_z - zp_y] = [L_z, y]p_z - z[L_z, p_y] = -i\hbar p_z + i\hbar z p_x = i\hbar L_y
\]  

(9)

\[
[L_z, r^2] = [L_z, x^2 + y^2 + z^2] = [L_z, x]x + x[L_z, x] + [L_z, y]y + y[L_z, y]
\]

\[
= 2i\hbar xy - 2i\hbar xy = 0
\]  

(10)

\[
[L_z, p^2] = [L_z, p_x^2 + p_y^2 + p_z^2] = [L_z, p_x]p_x + p_x[L_z, p_x] + [L_z, p_y]p_y + p_y[L_z, p_y]
\]

\[
= 2i\hbar p_x p_y - 2i\hbar p_x p_y = 0
\]  

(11)

\[
[L_z, H] = [L_z, \frac{p^2}{2m} + V] = \frac{1}{2m} [L_z, p^2] - [L_z, V(r)] = 0
\]  

(12)

Because of spherical symmetry,

\[
[L_z, H] = [L_y, H] = [L_z, H] = 0
\]  

(13)
3 4.23

(a) According to 3.73,
\[
\frac{d}{dt} \langle L \rangle = \frac{i}{\hbar} \langle [\hat{H}, L] \rangle + \frac{1}{\hbar} \langle \frac{\partial L}{\partial t} \rangle = \frac{i}{2m\hbar} \langle [p^2, L] \rangle + \langle [V(r), r \times \nabla] \rangle
\]
\[
= \frac{i}{2m\hbar} \langle [p^2, L] \rangle + \langle [V(r), r \times \nabla] \rangle
\] (14)

\[
[V(r), r \times \nabla] \Psi(r) = V(r)r \times \nabla \Psi(r) - r \times \nabla [V(r) \Psi(r)]
\]
\[
= V(r)r \times \nabla \Psi(r) - r \times (\nabla V) \Psi(r) - V(r)r \times \nabla \Psi(r)
\]
\[
= -r \times (\nabla V) \Psi(r) = N \Psi(r)
\]
\[
\therefore \frac{d}{dt} \langle L \rangle = \langle N \rangle
\] (15)

(b) From 4.19, we know for spherically symmetric potential,
\[
[L_x, H] = [L_y, H] = [L_z, H] = 0
\] (17)

So,
\[
\frac{d}{dt} \langle L \rangle = \frac{i}{\hbar} \langle [\hat{H}, L] \rangle = 0
\] (18)

Alternatively, with no angular dependence in V, \(\nabla V\) would be along \(r\), making \(N=0\).

4 4.29

(a).
\[
S_x = \frac{\hbar}{2} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\] (19)
\begin{align*}
\mathbf{S}_x, \mathbf{S}_y &= \frac{\hbar^2}{4} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\
&= \frac{\hbar^2}{4} \left[ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] \\
&= i\hbar \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar \mathbf{S}_z 
\end{align*}

\begin{align*}
\mathbf{S}_y, \mathbf{S}_z &= \frac{\hbar^2}{4} \left[ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] \\
&= \frac{\hbar^2}{4} \left[ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\
&= i\hbar \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\hbar \mathbf{S}_x 
\end{align*}

\begin{align*}
\mathbf{S}_z, \mathbf{S}_x &= \frac{\hbar^2}{4} \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\
&= \frac{\hbar^2}{4} \left[ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \\
&= i\hbar \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\hbar \mathbf{S}_y 
\end{align*}

(b).

\begin{align*}
\sigma_x^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 
\end{align*}

\begin{align*}
\sigma_y^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 
\end{align*}
\[ \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \quad (25) \]

\[ \sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \sigma_z \quad (26) \]

\[ \sigma_y \sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \sigma_x \quad (27) \]

\[ \sigma_z \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \sigma_y \quad (28) \]

Similarly, \(\sigma_y \sigma_x\) etc and the formula can be written together as:

\[ \sigma_j \sigma_k = \delta_{jk} + i \sum_l \varepsilon_{jkl} \sigma_l \quad (29) \]

### 5 4.30

(a).

\[ 1 = \chi^* \chi = A^2 \begin{pmatrix} -3i & 4 \\ 3i & 4 \end{pmatrix} = 25A^2 \quad (30) \]

\[ \therefore A = \frac{1}{5} \quad (31) \]

(b).

\[ \langle S_x \rangle = \chi^* S_x \chi = \frac{1}{25} \begin{pmatrix} -3i & 4 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \]

\[ = \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3i \end{pmatrix} = 0 \quad (32) \]
\begin{align*}
\langle S_y \rangle &= \chi^* S_x \chi = \frac{1}{25} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \left( \frac{\hbar}{2} \begin{pmatrix} 3i \\ i \end{pmatrix} \right) \\
&= \frac{\hbar}{50} \begin{pmatrix} -3i \\ 4 \end{pmatrix} \begin{pmatrix} -4i \\ -3 \end{pmatrix} = -\frac{12\hbar}{25} \tag{33}
\end{align*}

\begin{align*}
\langle S_z \rangle &= \chi^* S_x \chi = \frac{1}{25} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \left( \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \right) \\
&= \frac{\hbar}{50} \begin{pmatrix} -3i \\ 4 \end{pmatrix} \begin{pmatrix} 3i \\ -4 \end{pmatrix} = -\frac{7\hbar}{50} \tag{34}
\end{align*}

(c).

\begin{align*}
S_x^2 &= S_y^2 = S_z^2 = \frac{\hbar^2}{4} = \langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle \tag{35}
\end{align*}

\begin{align*}
\sigma_{S_x} &= \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2} \tag{36}
\end{align*}

\begin{align*}
\sigma_{S_y} &= \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2} = \sqrt{\frac{\hbar^2}{4} - \frac{(12\hbar/25)^2}{25}} = \frac{7\hbar}{50} \tag{37}
\end{align*}

\begin{align*}
\sigma_{S_z} &= \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2} = \sqrt{\frac{\hbar^2}{4} - \frac{(7\hbar/50)^2}{25}} = \frac{12\hbar}{25} \tag{38}
\end{align*}

(d).

\begin{align*}
\sigma_{S_x} \sigma_{S_y} &= \frac{\hbar}{2} \frac{7\hbar}{25} = \frac{\hbar}{2} |\langle S_z \rangle| \text{(at the limit)} \tag{39}
\end{align*}

\begin{align*}
\sigma_{S_y} \sigma_{S_z} &= \frac{7\hbar}{25} \frac{12\hbar}{25} = \frac{84\hbar^2}{625} > \frac{\hbar}{2} |\langle S_x \rangle| \tag{40}
\end{align*}

\begin{align*}
\sigma_{S_z} \sigma_{S_x} &= \frac{12\hbar}{25} \frac{\hbar}{2} = \frac{\hbar}{2} |\langle S_y \rangle| \text{(at the limit)} \tag{41}
\end{align*}