Total credit 40 points. Do all problems and show all work. This is a closed book/notes exam, but a one-sided 8.5x11 sheet with only formulae is allowed. Submit the sheet with your bluebook. No calculators/phones, tackle any (simple) calculations on your own.

1. (5 pts) True or false?
   i) Stern-Gerlach experiment (which showed spin quantization) employs a uniform magnetic field. False: The field has to be non-uniform.
   ii) When an extremely strong magnetic field is applied to the hydrogen atom, \( j \) is no longer a good quantum number. True: Torque due to the magnetic field destroys \( J \) conservation.
   iii) Degeneracy pressure of neutrons prevents a white dwarf from collapsing further. False: that from electrons.
   iv) Band gaps are a general consequence of a periodic potential. True.
   v) In quantum statistical mechanics, the total energy constraint results in the concept of temperature. True.

2. (8 pts) Starting with the Schrodinger equation, derive the stationary-state wavefunctions and energies for a particle of mass \( m \) in an infinite cubical well of dimension \( a \). Show your work step by step, but you don’t need to normalize the wavefunctions. Find the degeneracies for the lowest five energy levels.

   Soln: Separate the 3-d Cartesian Schrodinger equation by writing \( \psi(x,y,z) = X(x)Y(y)Z(z) \). Solve the separated equations to get the sinusoidal solutions \( \sin(n_x \pi x/a) \), etc, as well as the energy quantization condition \( E = \left( \frac{\hbar^2 \pi^2}{2ma^2} \right)(n_x^2 + n_y^2 + n_z^2) \), where \( n_i = 1,2,3... \) Now count the degeneracies as follows:

   Lowest: \( \Sigma n^2 = 3 \) : \( (n_x,n_y,n_z) = (111) \implies n_d = 1 \). Next: \( \Sigma n^2 = 6 \) : \( (211), (121), (112), n_d = 3 \). Next: \( \Sigma n^2 = 9 \) : \( (221), (212), (122), n_d = 3 \). Next: \( \Sigma n^2 = 11 \) : \( (311), (131), (113), n_d = 3 \). Next: \( \Sigma n^2 = 12 \) : \( (222), n_d = 1 \).

3. (8 pts) Use the variational principle to find an upper bound on the ground state energy for the quartic potential \( V(x) = x^4 \) using a gaussian trial wave function \( e^{-bx^2} \). Be sure
to normalize first. \(\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\), \(\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \sqrt{\pi}/2\), and \(\int_{-\infty}^{\infty} x^4 e^{-x^2} dx = 3\sqrt{\pi}/4\).

Sohn: Get the normalized trial function by integrating the square of given function: \(\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}\). Then,

\[
\langle V \rangle = \left(\frac{2b}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx \ x^4 \ e^{-2bx^2} = 2\sqrt{\frac{2b}{\pi}} \int_{0}^{\infty} dx x^4 e^{-2bx^2} = \frac{3}{16b^2}.
\]  

(1)

Similarly,

\[
\langle KE \rangle = \left(\frac{2b}{\pi}\right)^{1/2} \left(\frac{-h^2}{2m}\right) \int_{-\infty}^{\infty} dx \ e^{-bx^2} \left(d^2/\text{dx}^2\right) e^{-bx^2} = \ldots = \frac{h^2b}{2m}.
\]  

(2)

So

\[
\langle H \rangle = \frac{h^2b}{2m} + \frac{3}{16b^2}
\]  

(3)

\[
\frac{d}{db} \langle H \rangle = \frac{h^2}{2m} - \frac{3}{8b^3} = 0 \Rightarrow b = \left(\frac{3m}{4h^2}\right)^{1/3}
\]  

(4)

An upper bound of the ground state energy is

\[
\langle H \rangle_{\text{min}} = \frac{3}{2} \left(\frac{3h^4}{32m^2}\right)^{1/3}
\]  

(5)

4. (9 pts) (a) Derive the lowest order correction (\(\langle \psi_n | \delta H | \psi_n \rangle\)) to the energy levels of the one-dimensional harmonic oscillator (\(V = (1/2)m\omega^2 x^2\)) due to the perturbation \(\delta H = \alpha p^2/(2m)\), where \(\alpha \approx 0\). (b) Obtain the exact energy levels by recognizing that the perturbed Hamiltonian represents an oscillator with different \(\omega\). (c) Show that (a) follows from (b). \(p = i\sqrt{h\omega/2}(a_+ - a_-)\), \(a_+ \psi_n = \sqrt{n+1} \psi_{n+1}\), and \(a_- \psi_n = \sqrt{n} \psi_{n-1}\).

Sohn: (a) \(p^2 = -(h\omega/2)(a_+ - a_-)^2 \Rightarrow \delta H = (ah\omega/4)(a_-a_+ + a_+a_- + \text{irrelevant terms}) \Rightarrow \langle \psi_n | \delta H | \psi_n \rangle = (ah\omega/4)(\sqrt{n+1}\sqrt{n+1} + \sqrt{n}\sqrt{(n-1)+1}) = (\alpha/2)(n + 1/2)h\omega\).

Therefore, the first order corrected energies are \((1 + \alpha/2)(n + 1/2)h\omega\). (b) The perturbed Hamiltonian applies exactly to an oscillator with mass \(M = m/(1 + \alpha)\) and potential \((1/2)M(\sqrt{1+\alpha})^2 \omega^2 x^2\), thus with (exact) energies \((n + 1/2)h(\sqrt{1+\alpha}) \omega\). (c) Approximating \(\sqrt{1+\alpha}\) by \(1 + \alpha/2\) gives (a) from (b).

5. (10 pts) [There is little computation in this question, especially after part (a). You may assume \(\hbar = 1\) for convenience. \(\langle s \ m \rangle\) denotes the conventional eigenstates of \(S_z\).] A spin-2 particle is in the unnormalized \(S_x\) eigenstate \(\psi = |2 \ 2\rangle + 2|2 \ 1\rangle + \sqrt{6}|2 \ 0\rangle + 2|2 \ -1\rangle + |2 \ -2\rangle\).

(a) Use \(S_x = (S_+ + S_-)/2\), and \(S_{\pm}|2 \ m\rangle = \sqrt{2(2+1)-m(m+1)} |2 \ m \pm 1\rangle\) to find the \(S_x\)
eigenvalue

Soh: $S_+|\psi\rangle = 0 + 2\sqrt{6-2}|2\ 2\rangle + \sqrt{6}\sqrt{6}|2\ 1\rangle + 2\sqrt{6}|2\ 0\rangle + \sqrt{6} - (-2)(-1)|2 - 1\rangle$.

$S_-|\psi\rangle = \sqrt{6 - 2}|2\ 1\rangle + 2\sqrt{6}|2\ 0\rangle + \sqrt{6}\sqrt{6}|2\ - 1\rangle + 2\sqrt{6} - (-1)(-2)|2 - 2\rangle + 0$.

Therefore, $S_x|\psi\rangle = (S_+|\psi\rangle + S_-|\psi\rangle)/2 = .... = 2|\psi\rangle$, i.e., $S_x$ eigenvalue is 2.

(b) Describe precisely the results of measuring (i) $S_x$, (ii) $S_x^2$, (iii) $S^2$ and (iv) $S_y^2 + S_z^2$.

Soh: Always i)2, ii)4, iii)2(2+1)=6, iv)6-4=2

(c) Using b(iv), what is $\langle S_x^2 \rangle$?

Soh: Half of b(iv), i.e. 1

(d) (Normalize the wavefunction.) What are the possible measurement values for $S_z$, and what are their measurement probabilities?

Soh: Normalizing, $|\psi\rangle = |2\ 2\rangle/4 + |2\ 1\rangle/2 + \sqrt{3}/8|2\ 0\rangle + |2 - 1\rangle/2 + |2 - 2\rangle/4$. Therefore, the possible and measurement values(probabilities) are: +2(1/16), +1(1/4), 0(3/8), -1(1/4), and -2(1/16).

(e) What are the possible measurement values for $S_z^2$, and what are their measurement probabilities?

Soh: 4(1/8), 1(1/2), and 0(3/8).

(f) Using (d), calculate $\langle S_z \rangle$.

Soh: Since the positive and negative values occur with equal probability, the weighted average is zero. $\langle S_z \rangle = 0$.

(g) Using (e), calculate $\langle S_z^2 \rangle$.

Soh: $\langle S_z^2 \rangle = 4(1/8) + 1(1/2) + 0 = 1$.

(h) How can the maximum $S_z^2$ (=-4, from (e)) be larger than $S_y^2 + S_z^2$ (=-2, from (b))?

Soh: Given state is not an $S_z$ eigenstate. However, when a measurement results in, say, $S_z=2$ and hence $S_z^2=4$, the state collapses into the corresponding $S_z$ eigenstate, which is different from the given state. QM can be strange.