

# AAPT talk, Jan. 23, 2002

Analyzing Student Equation Entries in a Computer Tutorial System

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1  
Title

Thank you.

I am going to discuss work I did last year in building the algebra-handling subsystem of an intelligent tutoring system which is designed to help students with homework problems in introductory physics.

2  
ITS

An intelligent tutoring system, or ITS, is a computer program which interacts in a meaningful way with the student to further learning. Solving homework problems is one important component in the learning experience in quantitative physics courses. The ITS that I will discuss, Andes II, is designed to improve this experience.

An ITS is not a homework grader in the spirit of WebAssign, UT, Capa, and similar web-based homework systems, because an ITS is designed to act as a **tutor** giving active help to the student as she tries to do the problems. This requires that the tutor has a model of where the student is at in trying to solve the problem.

3  
Andes II  
picture

This model needs to be based on the steps taken so far by the student in tackling the problem. I will briefly describe what these are for Andes, a system developed at the University of Pittsburgh and the U. S. Naval Academy.

4  
Andes  
components

The Andes ITS consists of a number of components which work together, first in readying a problem for the tutor and then in tutoring the student. The first component is the **problem solver**, which takes a stylized problem statement and generates methods of solution. It begins by applying physical principles to generate a list of the “canonical equations” needed to solve the problem. From this list, the algebra system finds the solution to the equations generated, by which I mean the values of all the quantities involved in the problem. In particular that includes the answers to the questions posed by the problem. The problem solver then seeks solution paths by starting from the sought quantities and, for each equation which appears to help solve for that quantity, asking the **algebra system** whether it **in fact** advances the solution.

The tutoring itself is done by the **workbench** and the **help system**, in interaction with the algebra system. These handle communication with the student, including giving right/wrong feedback, explanations of what might be wrong with what the student did, and “next step” help when the student is stuck.

My work on Andes was on the algebra system, so I will focus only on that part. I am going to discuss an interesting philosophical issue that arose unexpectedly concerning what it means for a student’s equation to be correct. I will also discuss how to use correct student entries to model the student’s knowledge.

5 Opening problem
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First I want to orient you to the Andes input screen. At the beginning of a problem, the problem statement occurs in the upper left. This area is also used by the student for drawing forces and other vectors, and for defining axes. The student must define all variables used, and these definitions are catalogued on the upper right. Equations are entered on the lower right, and dialog occurs when appropriate on the lower left.

6  
Mid-  
solution

As the student defines variables, draws vectors, and inputs equations, each entry is marked green or red for correct or not, and dialogue is available when mistakes are made. Unfortunately we don't have time to discuss that, so I have only made correct entries. I am going to focus on an equation the student has written for Newton's second law applied to the hanging mass:

7  
Eqn for  
hanging m

The variables  $ab$  and  $Ftb$  represent the magnitudes of the acceleration and tension vectors, so a student might very reasonably write down the equation shown, which we will use as an example to illustrate why equation identification is non-trivial.

8  
Eqns hard  
to ID

First I will prettify the notation a little. The student has written an equation for Newton's second law on the hanging mass.

$$mg - T = ma$$

It is a perfectly reasonable equation for the student to write, but it is not, in fact, Newton's second law, which deals with vector forces and accelerations, and has nothing to do with the acceleration of gravity and does not deal directly with the magnitudes of vectors. The closest canonical equation, which **is** part of the application of Newton's second to the hanging mass, is

$$W_y + T_y = ma_y.$$

The student has, in fact, incorporated seven other canonical equations,

$$\begin{aligned} a_y &= a \sin \theta_a, & W_y &= W \sin \theta_W, & T_y &= T \sin \theta_T \\ \theta_a &= 270^\circ, & \theta_W &= 270^\circ, & \theta_T &= 90^\circ \\ W &= mg \end{aligned}$$

which include vector projection methods, the implicitly given values of the angles, and a general physical law. In order to properly track where the student is in attacking this problem, the help system must recognize that the one equation the student wrote down indicates awareness of all eight canonical equations.

So the issues I will address today are how to determine the correctness of equations and how to determine which canonical equations must have been known for the student to have written down what she did.

9 make list
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One approach is to make a list of all the equations the computer can derive by manipulating the canonical equations. While it does so, it keeps track of all canonical equations used. If the computer can make all correct derivations, the correctness of a student's equation is determined by whether or not it is on the list. The possible dependencies are determined by the computer's notes about what it did to derive the equation.

10 cons to list
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Of course to do this we need a computer algorithm which can make all the algebraic combinations of the canonical equations that are valid. It wouldn't surprise me to find that any set of methods sufficiently robust to generate all acceptable equations would, in moderately complex problems, run off with an infinite list of output equations. Indeed, this list-generation method is the approach tried by the earlier version of Andes, and for many straightforward problems the computers crashed before finishing the generation of the list. Thus this list-making method has proven unworkable.

11 color by numbers
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There is an alternative: the correctness of equations can be judged by whether they are true in context. That is, the solution to the problem, meaning the values for all the variables, is plugged in, and the equation is

correct if it balances. Once the equations for a problem have been solved, checking a student equation in this way is nearly instantaneous. But it gives no hint of dependencies. For that we must ask, for each subset of canonical equations, whether the student's equation is true whenever the equations in the subset are. For linear equations, this is merely a matter of evaluating the rank of a matrix, which is still a fast calculation. For nonlinear equations the issue would be much more complex, even unsolvable in general. But we can get a pretty good approximation if we ask whether the linear expansion of the student's equation around the solution point is independent of the linear expansions of the canonical equations.

<p>12 pros and cons</p>
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This method is very efficient — it is able to answer correctness instantaneously and dependency on many subsets in a time unnoticed by the student. But it does have two weaknesses — one of which is an unexpected surprise: while it correctly judges algebraic correctness, that might not be really what we want. The other problem is that the linearized equations can be dependent even when the full equations are not. This occurs only if the solution point happens to be at a critical value of the equations, but that is the case unexpectedly often. Fortunately, this occurs in a way which can be circumvented, but I haven't time to discuss that. Of more general interest, however, is the realization that algebraic correctness may not be what we want. Let me illustrate.

<p>13 pedagogic/ algebraic</p>
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One equation students are expected to be able to use in problems involving one-dimensional kinematics with constant acceleration is

$$v_f^2 - v_i^2 = 2as. \tag{1}$$

In many problems, the object is stated to be initially at rest, so another

canonical equation for such a problem is

$$v_i = 0 \tag{2}$$

Suppose a student writes down

$$v_f^2 + v_i^2 = 2as. \tag{3}$$

Doubtless the student has made a mistake with the sign of the  $v_i$  term. Color by numbers, however, would call this correct, as adding zero is equivalent to subtracting it. Eq. 3 is also derivable, as squaring Eq. (2), doubling it, and adding it to Eq. (1) derives the student's equation in an algebraically correct fashion.

14 pedagogic $\neq$ algebraic
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Thus while color-by-numbers accurately judges algebraic correctness, it might not be exactly what we want. What is the distinction? The operations we just used have no motivation except to justify the pedagogically incorrect equation. Better results might come from demanding **motivated derivability** rather than **algebraic derivability**!

There are some open questions here:

*What exactly does that mean?*

*How can it be efficiently implemented?*

15 references
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These are questions for future research. In particular, we need to have empirical tests of how often wrong choices are made by color-by-numbers and what the pedagogical consequences are. This should be compared to possible methods of compiling lists of derived equations, which may also make wrong choices, because algorithms sufficiently limited to handle complex problems without blowing up may miss perfectly reasonable student inputs.

I have listed some web sites from which you can get further information on Andes in general, and a preprint of my paper on this subject, as well as my email address. I would be happy to hear from you.

Thank you for your attention.