Physics 618 Seventeenth Lecture March 24, 2017

We have been working on the machinery for finding irreducible representations of SU(N) and SO(N) in terms of direct products of the defining representation. This required us first to build apparatus for the permutation group S_k acting on the indices of a tensor for the coefficients of an element of $\otimes_1^k \underline{\mathbf{N}}$. We first defined the *group algebra*, and discussed how elements of the group algebra could project out individual irreducible representations of S_k. Noting there is a representation η for each partition of k, we introduced the *Young graph*, an array of boxes, and then a *Young tableau* τ, a graph with the numbers $1 \ldots k$ in the boxes. For each tableau, we defined a symmetrization operator P_τ, an antisymmetrization operator Q_τ, and a Young operator. We also defined different linear combinations,

$$e_{ij}^\eta = \frac{\ell_\eta}{k!} \sum_{P \in S_k} \Gamma_{ji}^\eta(P^{-1})P,$$

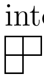
where Γ_{ji}^η is the irreducible representation of S_k. We saw that the set of e_{ij}^η for a fixed η form a two-sided ideal in the group algebra, and in fact give a decompositon of the identity

$$\mathbb{I} = \sum_{\eta\, i} e_{ii}^\eta$$

We also claimed that the e_{ij}^η were linear combinations of the form $e_{ij}^\eta = \sum_{ij} s_{ij} Q_i P_j$, where i and j are *standard tableaux* which have their box numbers increase left to right in each row and top to bottom in each column. These are fairly easy to count and give the dimension of the representation η, though for larger representations the counting is less easy, so it helps to have the magic formula

$$\ell_\eta = \frac{k!}{\prod_b g_b},$$

where g_b is the *hook* for box b.

Having found projectors for the irreducible representations of S_k, we turn to considering how these act on the tensors of SU(N) with k indices. In total, of course, there are N^k parameters, but these can be decomposed by the e_{ii}^η into directions which don't mix under SU(N). We saw, for example, that the  acting on a tensor component with two equal and one other index projects out one state from the three possible orderings, while it kills a state with all indices equal. In particular, we say that for SU(2), $\underline{\mathbf{2}} \otimes \underline{\mathbf{2}} \otimes \underline{\mathbf{2}} = \underline{\mathbf{4}} \oplus \underline{\mathbf{2}} \oplus \underline{\mathbf{2}}$.

Today

We will now formalize what we did in the simple example to discuss general N and k and a general η. We consider the subspace of states in $\otimes_1^k \underline{\mathbf{N}}$ with r_i indices equal to i, and note that while S_k will generate a span of this space from one ordering e, this need not give $k!$ independent vectors, because there is a subgroup of S_k which leaves the tensor unchanged, namely $\mathcal{P} = \otimes S_{r_i}$. Thus the space is also spanned by $\{P P_{\mathcal{P}} e | P \in S_k\}$, where $P_{\mathcal{P}}$ is the symmetrizer for $\mathcal{P}$. We will see that the dimension of the space projected out by e^η_{ii} from all permutations of e is the number of times the identity representation of $\mathcal{P}$ is contained in Γ^η, which is just

$$\gamma_\eta = \sum_{B \in \mathcal{P}} \chi^\eta(B) \Bigg/ \prod r_i!.$$

Next, we need to count how many choices there are of the r_i indices equal to i for SU(N). That is, there are N possibilities for all indices equal, $N(N-1)$ choices for all equal except one (for $k \geq 3$), and $\binom{N}{k}$ for all unequal. Multiplying these by the number of states for each choice, γ_η and summing over all possibilities gives the dimension of the representation of $SU(N)$.

This is all a nontrivial calculation, so we will give another magic rule which gives answers easily, again without proof. First, the γ_η are given by counting the *permissible placements* of $1 \ldots N$ in the boxes of the Young graph. But actually to get the dimensions of the SU(N) representations, which is our primary goal, there is an even easier recipe. It involves rules for placing something analogous to a hook, but based on N, and then dividing by the product of hooks.