Welcome back.

Before the break we were working on finding the finite-dimensional representations of the compact simple Lie groups, which we observed could be extracted from tensor products of **fundamental representations** D^i. For SU(3) the easiest approach is to take the symmetric product of some number of quarks and the symmetric product of another number of antiquarks, and reduce the tensor product of these. But there is also another approach which generalizes nicely to SU(N) and SO(N), possible because all of the fundamental representations are found within direct products of the *defining* representation. But this requires extracting irreducible pieces by more complicated symmetrization constraints, which in turn requires us to know more about the irreducible representations of the permutation group S_n. This led us to define the *group algebra*, in particular for S_n, a formal linear combination of the permutations with complex coefficients.

Recall that every permutation in S_n can be written, essentially uniquely, as a product of disjoint cycles, with the length of these cycles forming a partition of the integer n, and each conjugacy class of S_n corresponds to one partition. For each partition $n = \sum_r m_r\, r$, with m_r repeats of the integer r, we draw a box diagram, called a *Young graph*, with m_r rows of length r, arranged with their left ends aligned and with each row no longer than the one above it. That is how we begin today.

Today

Our next step is to define *Young tableaux*, which are Young graphs with the numbers 1 to n inserted in some order. For each tableau τ there is a symmetrizing permutation P_τ, an antisymmetrizing permutation Q_τ, and a *Young operator* $Y_\tau = Q_\tau P_\tau$. There are also elements of the group algebra e^η_{ij} associated with each irreducible representation η associated with the Young graph τ. This connection is somewhat messy, but the e^η_{ij} have wonderful properties, with $\{e^\eta_{ij}, i, j \in 1 \ldots \ell_\eta\}$ for a fixed representation η forming a two-sided ideal in the group algebra, and also a decomposition of the identity,

$$1\!\!1 = \sum_{\eta,i} e^\eta_{ii}\,.$$

This means that we can apply this decomposition to a tensor with n indices and break it up into irreducible components for each η, with dimension η_ℓ.

Because the $\{e^\eta_{ij}\}$ and the Young operators each span the same space, they are linear combinations of each other, in a way given by what is called the *standard tableaux*, which are easy to count, giving the dimension of the

the representation η. We will learn (sadly without proof, see Schensted) a magic recipe for finding the dimension for nontrivial Young graphs.

Then we will turn to applying these e^{η}_{ii} to the tensors of SU(N) to generate the states of each irreducible representation. We will give a simple example for $\eta = \yng(2,1)$ for SU(2). Then we will begin to evaluate the dimension for arbitrary representations for any N.