Chapter 17

Poincaré and Susy

17.1 Poincaré Invariance

A great advance in physics came when it was realized that it was better to think about motion in empty space rather than starting with what happen in the "real world" with gravity pulling things downwards, friction slowing things down, etc. First, that the laws of physics should be the same at one point as at another, *i.e.* that they are translation invariant, and also from one day to another, *i.e.* that they are invariant under translations in time as well. Also they are direction independent - space is isotropic. Another great breakthrough was Galileo's recognition that two reference systems in relative motion at constant velocity have the same physics, so there is an underlying galilean invariance under spatial and time translations, rotations, and boosts. Thus we have a 10 parameter Lie group of symmetry transformations. Of course Einstein corrected some details, finding that the boosts affected time as well as space, in such a way as to leave invariant

$$(ds)^2 = (d\vec{x})^2 - c^2(dt)^2$$

between any two events, and in particular $(ds)^2 = 0$ for light in vacuum. Thus physics is invariant under Poincaré transformations

$$\begin{aligned} x^\mu \to x'^\mu &= a^\mu{}_\nu x^\nu + b^\mu, \\ \text{where} \quad \eta_{\mu\rho} a^\mu{}_\nu a^\rho{}_\sigma &= \eta_{\nu\sigma}. \end{aligned} \qquad \eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We expect states to be within a irreducible representation of the symmetry group. One way of labeling the representation is according to the values

of operators which commute with all the elements of the group. These are called Casimir ooperators, and by Schur's Lemma this must be a constant value on all the states within an irreducible representation. For angular momentum, $L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with each of the L_i, so is a fixed value for each irreducible representation of the SO(3) rotation group, and in fact for a spin ℓ representation it is $\ell(\ell+1)$.

For the Poincaré group we have the momenta, which are the generators of translations, P_μ, and the Lorentz generators $L^\mu{}_\nu$, which generate the Lorentz transformations. $e^{ib^\mu P_\mu}$ is the group element which gives $x^\mu \to x^\mu + b^\mu$, and clearly these commute with each other, $e^{ib_1^\mu P_\mu} e^{ib_2^\mu P_\mu} : x \to x^\mu + b_1^\mu + b_2^\mu$, so $[P_\mu, P_\nu] = 0$. But the Lorentz transformations, which are pseudo-rotations, do not commute with each other, or with P_μ,

$$\left[L^\mu{}_\nu, P_\rho\right] = -i\delta^\mu_\rho P_\nu + i\eta_{\nu\rho}\eta^{\mu\sigma}P_\sigma.$$

Things work a little more elegantly if we use $\eta_{\mu\nu}$ to lower upstairs indices and its inverse $\eta^{\mu\nu}$ to raise downstairs indices. As 4×4 matrices $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ look the same, but in this relativistic context we always contract an upper and a lower index (and the summation convention only permits that), so they are not interchangeable. Then we use $\eta_{\mu\rho}$ to lower the first index on $L^\rho{}_\nu$, $L_{\mu\nu} := \eta_{\mu\rho}L^\rho{}_\nu$, which makes it antisymmetric, $L_{\mu\nu} = -L_{\nu\mu}$, and then

$$\begin{aligned} [L_{\mu\nu}, P_\rho] &= -i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\mu & (17.1) \\ [L_{\mu\nu}, L_{\rho\sigma}] &= -i\eta_{\nu\rho}L_{\mu\sigma} + i\eta_{\nu\sigma}L_{\mu\rho} + i\eta_{\mu\rho}L_{\nu\sigma} - i\eta_{\mu\sigma}L_{\nu\rho} & (17.2) \end{aligned}$$

which, along with

$$[P_\mu, P_\nu] = 0$$

constitute the Poincaré Lie algebra. Note that the subalgebra generated by the P_μ's is an invariant subalgebra and is Abelian, so the Poincaré group is neither simple nor semisimple.

[Note: this is a rehash of p. 63, though there we used the opposite metric.]

As L is sort of a rotation, we might expect something like P^2 to be invariant, and indeed we define

$$P^2 := \eta^{\mu\nu}P_\mu P_\nu$$

$$\begin{aligned} \left[L_{\rho\sigma}, P^2\right] &= \eta^{\mu\nu}\left\{[L_{\rho\sigma}, P_\mu]\, P_\nu + P_\mu\,[L_{\rho\sigma}, P_\nu]\right\} \\ &= \eta^{\mu\nu}\left\{(-i\eta_{\rho\mu}P_\sigma + i\eta_{\sigma\mu}P_\rho)\, P_\nu - i\eta_{\rho\nu}P_\mu P_\sigma + i\eta_{\sigma\nu}P_\mu P_\rho\right\} \\ &= -i\delta^\nu_\rho P_\sigma P_\nu + i\delta^\nu_\sigma P_\rho P_\nu - i\delta^\mu_\rho P_\mu P_\sigma + i\delta^\mu_\sigma P_\mu P_\rho \\ &= -iP_\sigma P_\rho + iP_\rho P_\sigma - iP_\rho P_\sigma + iP_\sigma P_\rho = 0. \end{aligned}$$

So P^2 commutes with the whole algebra and is a Casimir invariant.

Note that neither the Lorentz subgroup, generated by just the $L^\mu{}_\nu$'s, nor the whole Poincaré group, is compact, and we would not expect the unitary irreducible representations to be finite dimensional. For the Lorentz group, however, we will be interested in finite-dimensional representations, as described below, though we will need to adjust for their non-unitarity.

Consider the Pauli-Lubanski 4-vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu L_{\rho\sigma},$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor (Levi-Civita) with $\epsilon^{0123} = 1$. W has an interpretation as the spin, because in involves a (pseudo)rotation $L_{\rho\sigma}$ in a plane orthogonal to the 4-momentum P_ν. Because of the ϵ the order is irrelevant, $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}L_{\rho\sigma}P_\nu$, and W^μ commutes with the momentum

$$[W^\mu, P_\tau] = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left[L_{\rho\sigma}, P_\tau\right]P_\nu = i\eta_{\rho\tau}\epsilon^{\mu\nu\rho\sigma}P_\sigma P_\nu = 0.$$

W^μ also "rotates", or rather transforms under Lorentz transformation, as a 4-vector ought:

$$[L_{\mu\nu}, W_\rho] = -i\eta_{\mu\rho}W_\nu + i\eta_{\nu\rho}W_\mu,$$

and its square is invariant,

$$[L_{\mu\nu}, W_\rho W^\rho] = 0.$$

Thus in an irreducible representation of the Poincaré algebra, even though it is in general infinite dimensional, all the states will have the same value for P^2 and W^2.

Consider a representation with $P^2 < 0$, so all P_μ are timelike vectors. Call $m^2 = -P^2 = (P^0)^2 - \vec{P}^{\,2}$. The commuting momentum generators may be chosen diagonal, so the basis states each have a definite P^μ. Assume[1] $P^0 > 0$. Then any such P^μ with $P^2 = -m^2$ can be acted upon by a Lorentz transformation to give $P^\mu = (m, 0, 0, 0)$. On this state, W^0 vanishes and

[1]We have been a little glib about which Lorentz group we are considering. If we are discussing the isochronous proper group, which is the connected component of the less restricted group, it preserves the direction of time and the handedness of spatial coordinate systems. It is only this connected subgroup which is known, so far, to be an exact symmetry of physics. The sign of P^0 is then an invariant when $P^2 < 0$, timelike. We restrict our attention to representations with $P^0 > 0$.

$W^j = \frac{m}{2}\epsilon^{jk\ell}L_{k\ell} = mL_j$, where L_j is the usual rotation generator about the j axis. So we see that $\vec{W}/m$ is a spin angular momentum, and $W^2/m^2 = s(s+1)$ gives the intrinsic spin of the particle.

Note the irreducible representation has states with each value of $\vec{P}$, so is infinite dimensional. But for a given $\vec{P}$ there are only $2s+1$ states.

For representations with $P^2 = 0$ or $P^2 > 0$, things are somewhat different. Take a course in relativistic quantum field theory for more on the former, and we won't talk about the latter, tachyons.

Any representation of the Poincaré group is a representation of the Lorentz group, and it will be useful to consider those, as we can find finite dimensional ones. We sometimes like to be old fashioned and distinguish space from time, so let's define

$$J_\ell = \frac{1}{2}\epsilon_{\ell jk}L_{jk}, \qquad K_j = L_{0j}.$$

We can find their commutators from (17.2),

$$\begin{aligned}
[J_j, J_k] &= i\epsilon_{jk\ell}J_\ell, && (17.3)\\
[J_j, K_k] &= i\epsilon_{jk\ell}K_\ell, && (17.4)\\
[K_j, K_k] &= -i\epsilon_{jk\ell}J_\ell. && (17.5)
\end{aligned}$$

Eqs. (17.3) and (17.4) are the usual commutation relations for the rotation operator with any vector, $[J_j, V_k] = i\epsilon_{jk\ell}V_\ell$. So $\vec{J}$ and $\vec{K}$ rotate as vectors ought to, under the action of the angular momentum generators $\vec{J}$. Eq. (17.5) is something else however, a somewhat surprising statement that Lorentz boosts do not commute but rather their commutator is a generator of a rotation.

The algebra of these six generators is simplified if we consider the complex linear combinations $L_{j\pm} := \frac{1}{2}(J_j \pm iK_j)$, which satisfy the commutators

$$\begin{aligned}
[L_{j+}, L_{k+}] &= \frac{i}{4}\,\epsilon_{jk\ell}\left(J_\ell + iK_\ell + iK_\ell + J_\ell\right) = i\epsilon_{jk\ell}L_{\ell+}\,,\\
[L_{j+}, L_{k-}] &= \frac{i}{4}\,\epsilon_{jk\ell}\left(J_\ell - iK_\ell + iK_\ell - J_\ell\right) = 0,\\
[L_{j-}, L_{k-}] &= \frac{i}{4}\,\epsilon_{jk\ell}\left(J_\ell - iK_\ell - iK_\ell + J_\ell\right) = i\epsilon_{jk\ell}L_{\ell-}\,. && (17.6)
\end{aligned}$$

Thus we have two sets of mutually commuting generators, so we can find the possible representations of fields by asking how they transform under each of the two independant algebras, each of which has the commutation relations of ordinary rotations, SO(3) or SU(2). We know the finite dimensional representations from our quantum mechanics course — they are labelled by a total spin which is a half integer. So we will have two spins, $s_\pm$ for $\vec{L}_\pm$, that will label a finite dimensional $(2s_+{+}1)(2s_-{+}1))$ representation of the Lorentz group, ψ_{s_+,s_-}. The two we will consider are $\psi_{\frac12,0}$ and $\psi_{0,\frac12}$. For $\psi_{\frac12,0}$, $\frac12(\vec J + i\vec K)\psi = \frac12\vec\sigma\psi$, $\frac12(\vec J - i\vec K)\psi = 0$, so $\vec J\psi = \frac12\vec\sigma\psi$, $\vec K\psi = \frac{-i}{2}\vec\sigma\psi$. Notice that for real Lorentz transformations, with the coefficients of $\vec J$ and $\vec K$ real, the coefficients of $\vec\sigma$ will not be real, and this is *not* a unitary representation. The Lorentz group is not compact, as the rapidity can go to infinity, so this should not be a surprise. That means that $\psi^\dagger\psi$ will not be a scalar under Lorentz boosts. What is a scalar is $\psi^\dagger\psi P_0 + \psi^\dagger\vec\sigma\psi\cdot\vec P$, where P_μ is the momentum. Similarly for $\psi_{0,\frac12}$ $\psi^\dagger_{0,\frac12}\psi_{0,\frac12}P_0 - \psi^\dagger_{0,\frac12}\vec\sigma\psi_{0,\frac12}\cdot\vec P$ is a scalar.

We have found two irreducible representations of the Lorentz algebra, which means two irreducible representations of the connected component of the Lorentz group, also known as the proper orthochronous Lorentz group, which does not include parity or time-reversal. Under parity, $\vec J$ is unchanged but $\vec K$ changes sign, so $s_+ \leftrightarrow s_-$, and to have a representation of the group including parity, we need the direct sum of $\psi_{\frac12,0}$ and $\psi_{0,\frac12}$, which is a four-dimensional representation $\psi = \begin{pmatrix}\psi_{0,\frac12}\\ \psi_{\frac12,0}\end{pmatrix}$. Define fixed matrices $\gamma^j = \begin{pmatrix}0_{2\times2} & \sigma_j\\ -\sigma_j & 0_{2\times2}\end{pmatrix}$, $\gamma^0 = \begin{pmatrix}0_{2\times2} & \mathbb{1}_{2\times2}\\ \mathbb{1}_{2\times2} & 0_{2\times2}\end{pmatrix}$. Then it turns out the free Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

and the Lagrangian density is

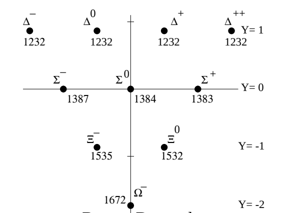
$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi.$$

where $\bar\psi := \psi^\dagger\gamma^0$.

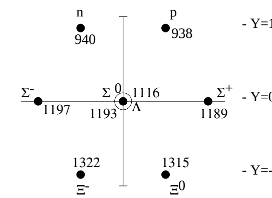
The representations $\psi_{\frac12,0}$ and $\psi_{0,\frac12}$ are known as Weyl spinors and the sum a Dirac spinor. Because the hermitian conjugate reverses the handedness, the hermitian lagrangian needs to have both right and left handed pieces. For the Dirac theory these are independent, but if we identify one with the hermitian conjugate of the other, we have a Majorana spinor. The Dirac spinor has 4 complex numbers, or 8 real ones, but the Majorana spinor has only 4 real numbers.

17.2 Supersymmetry

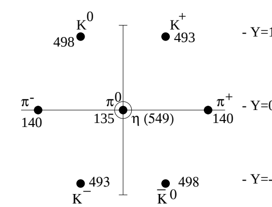
In the early days of flavor SU(3), it was noticed that the quarks could be considered as a sextuplet with three flavors and two spin states. If one imagined that the SU(2) rotational group on spin and the SU(3) flavor were parts of a unifying SU(6) acting on these six quark states, one naturally finds for three quarks the $\square\square\square$ a $\frac{6\cdot7\cdot8}{6} = 56$ dimensional representation which breaks into a **10** of SU(3) with spin 3/2 (40 states) and a flavor octet of spin 1/2, 16 states. These are precisely the low lying baryons. For $\begin{array}{l}\square\square\\ \square\\ \square\\ \square\end{array}$ appropriate for quark-antiquark mesons, one gets a **35** representation consisting of



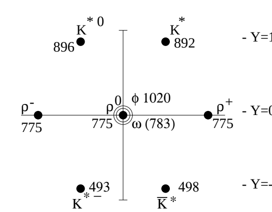
Baryon Decouplet



The Baryon Octet



pseudoscalar mesons



vector mesons

- a flavor octet with zero spin, which matches the three pi mesons, the 2 kaons and their two antiparticles, and a singlet called η. 8 states.
- a flavor octet with spin 1, matching the three ρ mesons, the four K^* and $\bar K^*$'s, and the neutral ω particle. 24 states.
- a flavor single spin 1 called the ϕ particle. 3 states.

This is an excellent match for the observed low lying mesons. In addition, SU(6) explained dipole moments of the baryons quite well.

The problem came from the theoretical argument that the rotation group is not an isolated symmetry which was free to unite with flavor, leaving behind the rest of the Poincaré group. All attempts to pull in the full group into a united whole led to inconsistencies, culminating in the

Coleman-Mandula Theorem:

The most general Lie algebra of symmetry generators which acts on particle states and has only finitely many particles of a given mass, and has a scattering matrix $S \neq 1$, consists of the Poincaré group, P_μ, $L_{\mu\nu}$, in direct product with an internal symmetry group with scalar charges Z_ℓ.

Thus SU(6) was dead! There is no Lie algebra which can combine internal symmetry and Poincaré symmetry in a nontrivial manner.

If that were the end of the story, I probably wouldn't be talking about it. But in fact independent developments took place in the late '60's, on a theory of elementary particles then called Dual Resonance Models, now known as string theory. In this theory, a new kind of infinitesimal symmetry generator was found, which mixed bosons and fermions! This was generalized to ordinary four dimensional field theory by J. Wess and B. Zumino, *e.g. Nucl. Phys.* **B70** (1974) 39.

Consider a Majorana fermion[2] ψ and two scalar fields A and B. Let α be a Majorana spinor which is not a field operator but almost a c-number, except that instead of commuting with everything, its components anticommute with themselves and other fermionic objects. Then under an infinitesimal transformation

$$\begin{aligned} \delta A &= i\bar{\alpha}\psi, \\ \delta B &= i\bar{\alpha}\gamma_5\psi, \\ \delta\psi &= \partial_\mu \left(A - \gamma_5 B\right)\gamma^\mu \alpha, \end{aligned}$$

the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A\right)\left(\partial^\mu A\right) - \frac{1}{2}\left(\partial_\mu B\right)\left(\partial^\mu B\right) - \frac{1}{2} i\bar{\psi}\gamma^\mu\partial_\mu\psi$$

is unchanged, that is, $\delta\mathcal{L} \sim 0$. Actually, it is not zero but a total derivative, but that is the same thing, because a total derivative in the Lagrangian density does not affect the equations of motion classically, and produces only a phase in the Feynman path integral quantum mechanically. This lagrangian density is the standard form for noninteracting massless scalars and a spinor

[2]In representations of the proper Poincaré group, there are two massive spin $\frac{1}{2}$ representations, one left-handed and one right-handed, which are, of course, related to each other by Parity. A Dirac fermion is the direct sum of these two representations, which are independent degrees of freedom of the field. For a Majorana fermion these are constrained: the right-handed piece is the hermitean conjugate of the left handed piece, and not independent. In Dirac language $\psi = C(\bar{\psi})^T$ where C is the charge-conjugation matrix, and $\bar{\psi} = \psi^\dagger\gamma^0$.

field. One could also add masses and interactions to $\mathcal{L}$ and still maintain the symmetry.

Now here is clearly a symmetry which enlarges the Poincaré group. The α_a (a a Dirac index) are parameters which must be associated with some operators Q_a which do the infinitesimal transformations, in the same way the Lie algebra generators do. The Grassmann (anticommuting) nature of the α's implies that the algebra of the Q's involves anticommutators rather than commutators. The anticommutators, being bilinears, must be ordinary symmetries of the Lagrangian. That is, they must form a Lie algebra. This means

$$\{Q_a, Q_b\} = A^\mu_{ab} P_\mu + B^{\mu\nu}_{ab} L_{\mu\nu} + C^j_{ab} Z_j$$

where Z_j are the generators of some internal symmetry Lie algebra, in accord with the Coleman-Mandula theorem.

The Q's must transform under the Lorentz group like spinors, and must be Majorana, $Q = C\bar{Q}^T$, where C is the charge conjugation[3] matrix $C^{-1}\gamma_\mu C = -\gamma_\mu^T$, and ^{T} means transpose (in the Dirac spinor space). So the index a can contain an internal index j as well as its Dirac index a. Then it turns out the only allowed possibility for massive particles is

$$\{Q_{ja}, Q_{kb}\} = -(\gamma^\mu C)_{ab} P_\mu \delta_{jk} + C_{ab} Z_{jk}. \tag{17.7}$$

The reference is R. Haag, J. T. Łopuszanski and M. Sohnius *Nucl. Phys.* **B88** (1975) 257.

This may be added to an internal symmetry algebra with generators B_m. The Q's are in some representation $\Gamma(B_m)$ of the internal symmetry algebra. The Z_{jk} need to commute with all the generators of these internal symmetries B_m, and also with all the other generators. They are called central charges, and therefore, if they exist, generate an invariant Abelian subalgebra. The

[3]Majorana is always a condition relating Q and $\bar{Q}$, but what the condition is depends on the representation chosen for the γ_μ's. C is then defined suitably.

full superalgebra then has the relations (17.7) and

$$\begin{aligned}
[B_m, B_n] &= ic_{mn}{}^p B_p \\
[L_{\mu\nu}, L_{\rho\sigma}] &= -i\eta_{\nu\rho}L_{\mu\sigma} + i\eta_{\nu\sigma}L_{\mu\rho} + i\eta_{\mu\rho}L_{\nu\sigma} - i\eta_{\mu\sigma}L_{\nu\rho} \\
[L_{\mu\nu}, P_\rho] &= -i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\mu \\
[P_\mu, P_\nu] &= 0 \\
[L_{\mu\nu}, B_m] &= [P_\mu, B_m] = 0 \qquad\qquad (17.8) \\
[B_m, Q_{ja}] &= -\Gamma_{jk}(B_m)Q_{ka} \\
[L_{\mu\nu}, Q_{ja}] &= \frac{1}{2}(\sigma_{\mu\nu})_{ab}Q_{jb} \\
[P_\mu, Q_{ja}] &= 0 \\
[Z_{jk}, B_m] &= 0 = [Z_{jk}, Z_{rs}] = [Z_{jk}, Q_{ja}] = [Z_{jk}, P_\mu] = [Z_{jk}, L_{\mu\nu}]
\end{aligned}$$

where Γ is a hermetean N dimensional representation of the group generated by the B's, and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ as usual in the Dirac algebra.

17.2.1 Superparticle Multiplets

Consider a single particle state. Its spin in the z direction is given by the operator $J_z = L_{xy}$ Thus

$$[J_z, Q_{ja}] = \frac{1}{2}(\sigma_{xy}Q)_{ja}, \qquad \text{where} \qquad \sigma_{xy} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}.$$

This shows that the Q_{1a} and Q_{3a} are raising operators by $\frac{1}{2}$ for J_z, while Q_{2a} and Q_{4a} lower J_z by 1/2.

There are many different representations of the Dirac algebra in common use. In our "Weyl" representation, $J_z = \frac{1}{2}\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix}$, $\gamma^0 = \begin{pmatrix}0&\mathbb{I}\\\mathbb{I}&0\end{pmatrix}$,

$$\gamma^i = \begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix}, \qquad C = \sigma^{02} = \begin{pmatrix}-i\sigma_2&0\\0&i\sigma_2\end{pmatrix} = \begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}.$$

As Q is Majorana,

$$Q = C\gamma_0^T Q^\dagger = \begin{pmatrix}0&-i\sigma_2\\i\sigma_2&0\end{pmatrix}Q^\dagger = \begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}Q^\dagger,$$

or $Q_1 = -Q_4^\dagger$, $Q_2 = Q_3^\dagger$.

Most of the interest in supersymmetry involves massless particles when the symmetry is unbroken. Consider a single massless particle state $|\psi\rangle$ with momentum $P^\mu = (E, 0, 0, E)$ moving along the z axis. Then $\gamma_\mu P^\mu = \begin{pmatrix}0&0&2&0\\0&0&0&0\\0&0&0&0\\0&2&0&0\end{pmatrix}E$, and $\{Q_{ja}, Q_{jb}\}|\psi\rangle = -2E\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}_{ab}|\psi\rangle$ if we assume no central charge. Now we see that $\{Q_{1j}, Q_{1j}^\dagger\} \neq 0$ acting on $|\psi\rangle$ and $Q_{1j}^\dagger$ is a lowering operator. Consider the state of highest $J_z = \lambda_{\max}$. Then $Q_1|\psi\rangle = 0$, but $\langle\psi|Q_1Q_1^\dagger|\psi\rangle = \langle\psi|\{Q_1, Q_1^\dagger\}|\psi\rangle = -\langle\psi|Q_1Q_4|\psi\rangle = 2E \neq 0$, so $Q_1^\dagger|\psi\rangle \neq 0$ and has a lower helicity.

If we have different j's, the Q_{1j}'s anticommute with each other and with the other $Q_{1k}^\dagger$'s (for $j \neq k$) even if we have central charges, because $(C)_{ab} = 0$ for $a, b = 1, 4$. Thus on the maximum helicity state all Q_{1j}'s vanish, while Q_{4j} can be applied once for each j, lowering the helicity by 1/2 each time.

Suppose we have N Q_{aj}'s, $j = 1, \ldots, N$. Suppose the maximum helicity state ψ has helicity λ. Then on this state we have the states shown in the table. This goes on until $\left(\prod_{i=1}^N Q_{4i}\right)|\psi\rangle$ produces just one state with helicity $\lambda - N/2$.

state	helicity	# of states
$\|\psi\rangle$	λ	1
$Q_{4j}\|\psi\rangle$	$\lambda - \frac{1}{2}$	N
$Q_{4j}Q_{4k}\|\psi\rangle$	$\lambda - 1$	$\binom{N}{2}$
$Q_{4j}Q_{4k}Q_{4\ell}\|\psi\rangle$	$\lambda - \frac{3}{2}$	$\binom{N}{3}$
$\vdots$	$\vdots$	$\vdots$
$\prod_{j=1}^N Q_{4j}\|\psi\rangle$	$\lambda - \frac{N}{2}$	1

One very exciting field of application of supersymmetry is supergravity. Supergravity theories involve the graviton, which means two massless states, of helicity ± 2. They also involve states of lesser helicity, but not states with $|\text{helicity}| > 2$, which cause grave problems in field theory when massless. This means that if we start with a graviton and work our way down, we will wind up with an impossible $\lambda = -5/2$ state if

$N > 8$. So the largest supergravity theory has 8 Q_j's. The massless states (before the required supersymmetry breaking) are given in the table.

helicity λ	name	# in $N = 8$
2	graviton	1
3/2	gravitino	8
1	gluons,$W^{\pm}, Z, \gamma, etc.$	28
1/2	quarks,leptons, *etc.*	56
0	Higgs?, *etc.*	70
$\vdots$	$\vdots$	$\vdots$
-2	graviton	1

The theory is very attractive in that it seems to be at least very close to being a finite field theory of gravity, which is much better than ordinary Einstein gravity coupled to matter (electrons, quarks, *etc.*). In quantum field theory, spin 2 particles lead to loop diagrams with nonrenormalizable infinities unless some symmetry forces them to vanish. Such symmetries do exist for gravity coupled only to itself, but not when coupled to "matter" fields. If, however, we have a supergravity multiplet, symmetries apply to the whole multiplet[4]. In the supermultiplet built on the graviton, could the massless states given by our discussion of the helicity multiplets give the "physical particles" at today's level of understanding? They should include states in multiplets of color $SU(3)_C \times$ Salam-Weinberg $SU(2) \times U(1)$:

needed particle	property under color	property under SW
graviton	$\mathbf{1}$	$\mathbf{1}$
gluons	$\mathbf{8}$	$\mathbf{1}$
photon, $W^{\pm}$, Z	$\mathbf{1}$	$\mathbf{3} \oplus \mathbf{1}$
quarks	$6 \times \mathbf{3}$	$3 \times \mathbf{2}$
leptons	$\mathbf{1}$	$3 \times \mathbf{2}$

There is an argument by Gell-Mann that O(8) supergravity is not quite big enough. There seem to be enough particles, but their properties are not quite right. For example, if the gluons are to be made by $Q_{4j}Q_{4k}\,|\text{grav}\rangle$, there must be some Q's which are $\mathbf{3}$ and some $\bar{\mathbf{3}}$ under color SU(3). These must be Salam-Weinberg singlets because there aren't enough Q's for them to be doubled. So Salam-Weinberg gauge fields, which are an adjoint representation

[4]Sugra + matter is not 1-loop finite, but pure sugra is 1- and 2-loop finite, but not 3-loop. Is extended sugra ($N = 8$ in particular) finite to all loops? Green, Russo and Vanhove, JHEP 0707:099 (2007)

of SU(2)×U(1), can only come from the remaining 2 Q's, antisymmetrized, while we need 4 colorless gauge particles. There could be no charged W's and no quarks which are both SU(3) $\mathbf{3}$ and SU(2) doublets. There is, nonetheless, still work being done on this idea[5].

17.3 Superfields

Reference: A. Salam and J. Strathdee, *Nucl. Phys.* **B76** (1974) 477, *Phys. Rev.* **D11** (1974) 1521.

In ordinary field theory, the momentum operator acts to generate, from a single operator $\phi(x^\mu = 0)$, the whole operator-valued field

$$\phi(x) = e^{iPx}\phi(0)e^{-iPx}.$$

Now we have an algebra which includes Q_m as well as P_μ. (From now on we specialize to $N = 1$ for simplicity, so $Q_{aj} \to Q_a$.) Let us define

$$\phi(x, \theta) = e^{iP_\mu x^\mu + i\bar{\theta}_a Q_a}\phi(0)e^{-iP_\mu x^\mu - i\bar{\theta}_a Q_a}$$

which is a function not only of x but also of the 4 Grassmann quantities θ_a. If we expand this function in a power series in θ, things will terminate with quartic terms because each $(\theta_a)^2 = 0$.

Unlike the P's. the Q's do not all (anti) commute. Thus

$$\begin{aligned}
\bar{\epsilon}\frac{\partial}{\partial\bar{\theta}}e^{\pm i\bar{\theta}Q} &= \int_0^1 e^{\pm i\alpha\bar{\theta}Q}(\pm i\bar{\epsilon}Q)e^{\pm i(1-\alpha)\bar{\theta}Q}\,d\alpha \\
&= \pm\int_0^1 e^{\pm i\alpha\bar{\theta}Q}(i\bar{\epsilon}Q)e^{\mp i\alpha\bar{\theta}Q}\,d\alpha\, e^{\pm i\bar{\theta}Q} \\
&= \pm\int_0^1 \left\{i\bar{\epsilon}Q \pm i^2\alpha\left[\bar{\theta}Q, \bar{\epsilon}Q\right]\right\}d\alpha\, e^{\pm i\bar{\theta}Q}
\end{aligned}$$

as[6] the commutator

$$\left[\bar{\theta}Q, \bar{\epsilon}Q\right] = -\bar{\theta}_b\bar{\epsilon}_a\{Q_b, Q_a\} = -\bar{\epsilon}\gamma_\mu C\bar{\theta}P^\mu = -\bar{\epsilon}\gamma_\mu\theta P^\mu,$$

[5]Krzysztof A. Meissner and Hermann Nicolai, "Standard Model Fermions and N=8 supergravity", arXiv:1412.1715v2 (2/23/15)

[6]By the Hadamard Lemma or Campbell-Baker-Hausdorff identity.

commutes with $\bar\theta Q$. Then

$$\begin{aligned}\bar\epsilon\frac{\partial}{\partial\bar\theta}e^{\pm i\bar\theta Q} &= \pm\left(i\bar\epsilon Q \pm \frac{1}{2}\bar\epsilon\gamma_\mu\theta P^\mu\right)e^{\pm i\bar\theta Q}\\ &= \pm e^{\pm i\bar\theta Q}\left(i\bar\epsilon Q \mp \frac{1}{2}\bar\epsilon\gamma_\mu\theta P^\mu\right)\end{aligned}$$

where the change of sign in the last expression comes from $[i\bar\epsilon Q, \pm i\bar\theta Q]$.

Thus

$$\begin{aligned}\bar\epsilon\frac{\partial}{\partial\bar\theta}\phi(x,\theta) &= \left(i\bar\epsilon Q + \frac{1}{2}\bar\epsilon\gamma_\mu\theta P^\mu\right)\phi(x,\theta) + \phi(x,\theta)\left(-i\bar\epsilon Q - \frac{1}{2}\bar\epsilon\gamma_\mu\theta P^\mu\right)\\ &= i\left[\bar\epsilon Q, \phi(x,\theta)\right] + \frac{1}{2}\bar\epsilon\gamma^\mu\theta\left[P_\mu, \phi(x,\theta)\right].\end{aligned}$$

As P_μ acts as the derivative ∂_μ on ϕ, we see that acting on ϕ,

$$\bar\epsilon Q = -i\bar\epsilon\left(\frac{\partial}{\partial\bar\theta} + \frac{i}{2}\gamma^\mu\theta\partial_\mu\right).$$

Let us write out the power series expansion. The linear terms in θ can be recombined into $\bar\theta$. The quadratic terms $[\theta_m, \theta_n]$ can be written as $\bar\theta M\theta$ for some Dirac matrix M. As the θ are Majorana, $\bar\theta = (C^{-1}\theta)^T = -\theta C^{-1}$ as C is antisymmetric, so $\bar\theta M\theta = -\theta C^{-1}M\theta = -\frac{1}{2}\theta_a\theta_b\left(C^{-1}M - (C^{-1}M)^T\right)_{ab}$. All 4×4 matrices are linear combinations of

$$\underbrace{\mathbb{I}}_{\mathrm{S}},\ \underbrace{\gamma_\mu}_{\mathrm{V}},\ \underbrace{\sigma_{\mu\nu}}_{\mathrm{T}},\ \underbrace{\gamma_\mu\gamma_5}_{\mathrm{A}},\ \underbrace{\gamma_5}_{\mathrm{P}}$$

where the labels stand for scalar, vector, tensor, axial vector, and pseudoscalar, which is what these matrices give if sandwiched between $\bar\psi$ and ψ. As the charge conjugation matrix satisfies $C^{-1}\gamma_\mu C = -\gamma_\mu^T$, $C^{-1}M$ is symmetric for V and T, and antisymmetric for S, P, and A. Only the antisymmetric ones can contribute to the terms quadratic in θ, so these terms are linear combinations of $\bar\theta\theta$, $\bar\theta\gamma_5\theta$ and $\bar\theta\gamma_\nu\gamma_5\theta$. There are 4 cubic terms which can be written in terms of $(\bar\theta\theta)\bar\theta$ and one quartic term $(\bar\theta\theta)^2$. Thus

$$\begin{aligned}\phi(x,\theta) &= A(x) + \bar\theta\psi(x) + \frac{1}{4}\bar\theta\theta F(x) - \frac{i}{4}\bar\theta\gamma_5\theta G(x) + \frac{1}{4}\bar\theta\gamma_\nu\gamma_5\theta A^\nu(x)\\ &\quad + \frac{1}{4}\bar\theta\theta\bar\theta\chi(x) + \frac{1}{32}(\bar\theta\theta)^2 D(x)\end{aligned}$$

expresses the superfield ϕ in terms of ordinary fields $A, \psi, F, G, A^\nu, \chi, D$

This superfield is not irreducible. Consider the operators $D_m = \dfrac{\partial}{\partial\bar\theta_m} - \dfrac{i}{2}(\gamma^\mu\theta)_m\,\partial_\mu$ which differ from the Q's only by the sign of the ∂_μ term. Now[7] $\{D_m, Q_n\} = 0$, so although the D's do not anticommute with each other, they do commute or anticommute appropriately with the P's and Q's. Under Lorentz transformations

$$[L_{\mu\nu}, D_m] = \frac{1}{2}\left(\sigma_{\mu\nu}D\right)_m.$$

From the four D's, let us divide into two pairs,

$$\begin{aligned}D_L &= \frac{1}{2}(1-\gamma_5)\,D,\\ D_R &= \frac{1}{2}(1+\gamma_5)\,D.\end{aligned}$$

Then the pairs do not mix under the supersymmetry transformations. Thus constraint equations such as $D_L\phi = 0$ are invariant under supersymmetries. A superfield satisfying such a constraint is called a right-handed **chiral superfield**.

We can solve this constraint conveniently if we note

$$\begin{aligned}D_L &= \frac{1}{2}(1-\gamma_5)\,D = \frac{1}{2}(1-\gamma_5)\left(\frac{\partial}{\partial\bar\theta_m} - \frac{i}{2}\gamma^\mu\theta\partial_\mu\right)\\ &= \frac{1}{2}(1-\gamma_5)\,e^{\frac{i}{4}\bar\theta\partial\!\!\!/\gamma_5\theta}\,\frac{\partial}{\partial\bar\theta}\,e^{-\frac{i}{4}\bar\theta\partial\!\!\!/\gamma_5\theta}\end{aligned}$$

$$\begin{aligned}\text{so}\quad D_L\phi = 0 &\implies (1-\gamma_5)\frac{\partial}{\partial\bar\theta}e^{-\frac{i}{4}\bar\theta\partial\!\!\!/\gamma_5\theta}\phi = 0\\ &\implies \phi = e^{\frac{i}{4}\bar\theta\partial\!\!\!/\gamma_5\theta}\,\tilde\phi \quad\text{where}\quad (1-\gamma_5)\frac{\partial}{\partial\bar\theta}\tilde\phi = 0.\end{aligned}$$

Although ϕ was real, $\tilde\phi$ is complex. However, the constraints require

$$\tilde\phi = A_-(x) + \bar\theta\psi_-(x) + \frac{1}{2}\bar\theta\frac{1+\gamma_5}{2}\theta F_-(x),$$

[7]To maintain the Grassmann character, we take $\partial/\partial\theta_m$ to satisfy $\{\partial/\partial\theta_m, \theta_n\} = \delta_{mn}$ instead of a commutator.

with no cubic or quartic terms, and with $(1-\gamma_5)\psi_-=0$, so ψ_- is a Weyl fermion field.

Consider the effect of a supersymmetry on ϕ:

$$\begin{aligned}
\bar\epsilon Q\phi &= -i\bar\epsilon\left(\frac{\partial}{\partial\bar\theta}+\frac{i}{2}\gamma^\mu\theta\partial_\mu\right)e^{\frac{i}{4}\bar\theta\not\partial\gamma_5\theta}\tilde\phi \\
&= e^{\frac{i}{4}\bar\theta\not\partial\gamma_5\theta}\left(-i\bar\epsilon\left[\frac{\partial}{\partial\bar\theta}+i\gamma^\mu\theta\partial_\mu\right]\right)\tilde\phi \\
&= e^{\frac{i}{4}\bar\theta\not\partial\gamma_5\theta}\left(\bar\epsilon\gamma^\mu\theta\partial_\mu\left[A_-+\bar\theta\psi_-+\frac{1}{2}\bar\theta\frac{1+\gamma_5}{2}\theta F_-\right]-i\bar\epsilon\left[\psi_-+\frac{1+\gamma_5}{2}\theta F_-\right]\right).
\end{aligned}$$

We know that $D_L\bar\epsilon Q\phi=\bar\epsilon QD_L\phi=0$, so $\bar\epsilon Q\phi$ must be a right handed chiral superfield of the form

$$\bar\epsilon Q\phi=e^{\frac{i}{4}\bar\theta\not\partial\gamma_5\theta}\left(\delta A_-+\bar\theta\delta\psi_-+\frac{1}{2}\bar\theta\frac{1+\gamma_5}{2}\theta\delta F_-\right).$$

It takes some considerable θ algebra, involving Fierz identities, but you can eventually show

$$\begin{aligned}
\delta A_- &= \bar\epsilon\psi_- \\
\delta\psi_- &= \frac{1+\gamma_5}{2}\left(F_--i\not\partial A_-\right)\epsilon \\
\delta F_- &= -i\bar\epsilon\not\partial\psi_-.
\end{aligned}$$

Notice that under a supersymmetry transformation, the highest θ term, here an F, changes by a total derivative. This will be the case for any superfield, because the term in $\bar\epsilon Q\phi=-i\bar\epsilon\left(\frac{\partial}{\partial\bar\theta}+\frac{i}{2}\gamma^\mu\theta\partial_\mu\right)\phi$ with the maximum number of θ's cannot get a contribution from $\frac{\partial}{\partial\bar\theta}\phi$, so it must be $\frac{1}{2}\bar\epsilon\gamma^\mu\theta\partial_\mu\phi$, a total derivative.

That means that if we take as a Lagrangian the highest θ term, either F in a chiral superfield or D in a general superfield, under supersymmetry transformations $\delta\mathcal{L}$ will be a total derivative and will contribute to the action only a surface integral.

Except for gauge fields, surface integrals at infinity can generally be taken to be zero, because physics should be local and the dynamics of fields in the laboratory should be independent of boundary conditions on the fields out

beyond Pluto. So effectively the variation of the action is zero, and the theory will be supersymmetry invariant.

Suppose our fundamental field is a chiral superfield ϕ. Then to make our Lagrangian we might use the F term of ϕ or ϕ^2 or ϕ^3, *etc.*, all of which are chiral superfields because $D_L(\phi_1\phi_2)=(D_L\phi_1)\phi_2+\phi_1(D_L\phi_2)=0$ if ϕ_1 and ϕ_2 are right-handed chiral superfields. The F terms are

of ϕ	F_-
of ϕ^2	$2A_-F_- - \frac{1}{2}\bar\psi_-(1+\gamma_5)\psi_-$
of ϕ^3	$3A_-F_- - 3A_-\bar\psi_-(1+\gamma_5)\psi_-$

so with ϕ^3 terms we get interaction terms and mass terms, but no kinetic energy terms. Note also that ϕ is not hermitean, because $\phi^\dagger$ is a left handed superfield. So $\mathcal{L}$ must involve $\phi^\dagger$ as well.

What about $\phi^\dagger\phi$? This is a product of left and right handed superfields and is therefore not chiral. So the Lagrangian must be taken from the D term, which turns out to be

$$\partial_\mu A_-^\dagger\partial^\mu A_-+\frac{i}{2}\bar\psi\not\partial\psi+F_-^\dagger F_-$$

where $\psi=\psi_-+\psi_-^\dagger$. This gives the usual kinetic energy terms for the scalar and spinor.

Consider just this term for $\mathcal{L}$

$$\mathcal{L}=-\frac{1}{2}\partial_\mu A_-^\dagger\partial^\mu A_- -\frac{i}{2}\bar\psi\not\partial\psi-F_-^\dagger F_-.$$

The "equations of motion" give free particle equations for A and ψ, but just a constraint equation for F, $F_-=0$. This is because there is no $\frac{\partial F}{\partial t}$ term in the Lagrangian. Fields like that are called subsidiary terms, and satisfy constraint equations rather than equations of motion, and do not correspond to particles. Thus the chiral superfield provides a supersymmetric theory including one scalar and one spinor particle.

Note this chiral multiplet has a complex scalar, equivalent to two real scalars, and one Majorana fermion. A Majorana fermion, with four real components to the spinor, is equivalent to a Weyl fermion, with two complex components, and consists of two physical states (either two helicities for a (Majorana) self-conjugate particle, or one each for a Weyl particle and its antiparticle. Thus the number of physical particles which are bosons and the number which are fermions are equal. This is counting "on shell" states. We can also count "off-shell" degrees of freedom, including with the complex A_- and F_- each counting as two real ones, and the chiral ψ_- counting as four.

In the full real superfield ϕ, each product of k θ's is multiplied by a totally antisymmetric tensor with k Majorana indices, for $\binom{4}{k}$ real off-shell degrees of freedom, so $1+6+1=8$ bosonic and $4+4=8$ fermionic ones. From this reducible multiplet we have extracted the left and right handed chiral multiplets, totaling $4+4$, so that leaves the "vector" superfield with 4 real bosonic fields and 4 real spinor components. The four fermionic ones are a Majorana fermion, and the four bosonic ones are a photon field, each of which have two on-shell states.

There is much more to be said about supersymmetry. For a fuller exposition of the fields, there is a standard text by Wess and Bagger, Supersymmetry and Supergravity, Princeton University Press (1992).