Chapter 10

Representations of Lie Groups

10.1 Fundamental Weights

Let us label the simple roots $\vec{\alpha}^i$, $i = 1, \ldots, m$. They and their conjugates generate (not linearly) any element of the Lie algebra. Thus any representation can be determined by how it behaves under these roots[1].

The weights have an ordering, so any finite dimensional representation has a greatest weight $\vec{\mu}_{\text{max}}$. The weight $\vec{\mu}$ of a basis element is $\vec{\mu}_{\text{max}}$ if and only if $\vec{\mu}+\vec{\alpha}^i$ is not a weight for each $\vec{\alpha}^i$. Define $q^i = 2\vec{\alpha}^i \cdot \vec{\mu}_{\text{max}}/(\alpha^i)^2$, which can be arbitrary nonnegative integers. As the $\vec{\alpha}^i$ are linearly independent and complete, the $\{q^i\}$ and $\vec{\mu}_{\text{max}}$ determine each other.

Define m vectors $\vec{\mu}^j$ such that

$$\frac{2\vec{\alpha}^i \cdot \vec{\mu}^j}{(\alpha^i)^2} = \delta_{ij}.$$

These are called the **fundamental weights** of the Lie algebra corresponding (as maximal weights) to a set of representations D^j called the **fundamental representations**

Warning: $\vec{\mu}^i$ is a vector, the i'th fundamental weight, while μ_i is the i'th component of any old weight $\vec{\mu}$. There is no connection, although each index takes on the same values $1, 2, \ldots, m$.

A tensor product of representations of a group has weights which are just the sum of the weights. In particular, the maximum weight of the

[1]A mathematician, who calls the representation the matrices which act on the vector space, and module that which the physicist calls the representation, would say instead the representation is determined by how it represents these roots.

product is the sum of the maximum weights of the factors. Thus we can find a representation of arbitrary maximal weight $\vec{\mu} = \sum q^i \vec{\mu}^{\,i}$ by reducing the tensor product

$$\bigotimes_i \left(D^i\right)^{q^i}.$$

10.1.1 SU(3) Multiplets

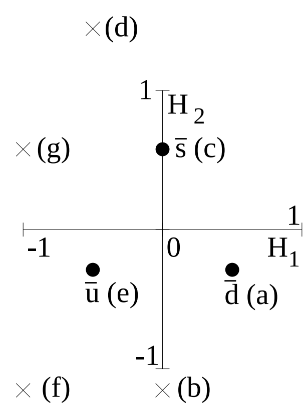
For SU(3), the positive roots are T_+, V_+, and U_-, of which the last two are simple.

$$V_+ = E_{1/2,\sqrt{3}/2}, \qquad U_- = E_{1/2,-\sqrt{3}/2},$$

$$\text{so} \quad \vec{\alpha}^{\,1} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \vec{\alpha}^{\,2} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \qquad T_+ = E_{\vec{\alpha}^{\,1}+\vec{\alpha}^{\,2}} = E_{(1,0)}.$$

The corresponding fundamental weights with $2\dfrac{\vec{\alpha}^{\,i}\cdot\vec{\mu}^{\,j}}{(\alpha^i)^2} = \delta_{ij}$ is solved by $\vec{\mu}^{\,1} = \left(\dfrac{1}{2}, \dfrac{1}{2\sqrt{3}}\right)$, $\vec{\mu}^{\,2} = \left(\dfrac{1}{2}, -\dfrac{1}{2\sqrt{3}}\right)$.

Let us generate the representation D^2. The q's are $q^1 = 0$, $q^2 = 1$, the highest weight state is $\bar{d} := \left|\vec{\mu}^{\,2}\right\rangle$ (a). Acting on this state $V_- = E_{-\vec{\alpha}^1}$ vanishes (b), and $E_{-\vec{\alpha}^2}\left|\vec{\mu}^{\,2}\right\rangle$ can act once, giving a non-zero state, proportional to $\bar{s} := \left|0, 1/\sqrt{3}\right\rangle$ (c). Now acting on this state again with $E_{-\vec{\alpha}^2}$ gives zero (d), as $2 > q^2 = 1$, but as $2\frac{\vec{\alpha}^{\,1}\cdot(0,1/\sqrt{3})}{(\alpha^1)^2} = 1$, the q for $\vec{\alpha}^{\,1}$ on $\bar{s}$ is 1, and so $E_{-\vec{\alpha}^1}\left|0, 1/\sqrt{3}\right\rangle \propto \bar{u} := \left|-\frac{1}{2}, -1/2\sqrt{3}\right\rangle$ (e) is not zero, but a second application vanishes rather than giving a state at $(-1, 2/\sqrt{3})$ (f).
We must still check $\vec{\alpha}^2$ on $\bar{u}$, but $\vec{\alpha}^{\,2}\cdot(-\frac{1}{2}, -1/2\sqrt{3}) = 0$ and we already know $p = 0$, so $q = 0$ and there is no state at $(-1, 1/\sqrt{3})$ (g).

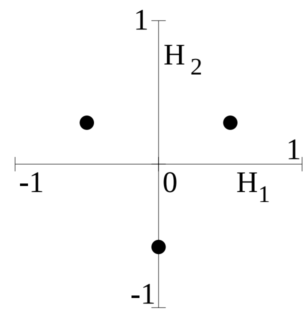


So we are done, having found three basis vectors, as there is no other way to get another state.

In flavor SU(3), these are the antiquarks. On the horizontal axis, $H_1 = T_3$ is the isospin component. The highest weight state (rightmost) is the anti-d quark $|\mu^2\rangle$, part of an isospin doublet with the anti-u quark $T_-\,|\mu^2\rangle =$

$E_{-\alpha^1}E_{-\alpha^2}\left|\mu^2\right\rangle$. The antistrange quark $\bar{s} = E_{-\alpha^2}\left|\mu^2\right\rangle$ is an isosinglet, $T = T_3 = 0$. The vertical axis is generally described in terms of strangeness or hypercharge.

Strangeness S is defined as zero for the u and d quarks and their antiparticles, and 1 for the anti-strange quark. Baryon number B is invariant under SU(3), and is defined as 1/3 the number of quarks minus the number of antiquarks, so is $-1/3$ for all the antiquark states. The (strong[2]) hypercharge $Y = S+B$ (in the absence of charm, topness and bottomness) is then 2/3 for $\bar{s}$ and $-1/3$ for $\bar{u}$ and $\bar{d}$. This representation is the conjugate of the representation D^1 corresponding to the fundamental weight μ^1, which is also called the **defining representation** for $SU(3)$. In flavor SU(3) this is the representation of the first three quarks, with the up quark u at the upper right, the down quark d to its left, and the strange quark s at the bottom.



The defining representation of SU(3)

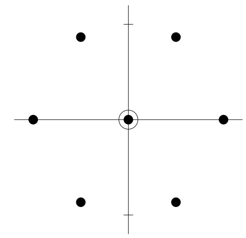
The quarks have $B = 1/3$. The electric charge is $Q = T_3 + Y/2$ times the positron charge e. Then the quarks have the quantum numbers as shown. The antiquarks have all the quantum numbers (except T) reversed.

quark	B	T	T_3	S	Y	Q/e
u	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{2}{3}$
d	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{3}$	$-\frac{1}{3}$
s	$\frac{1}{3}$	0	0	-1	$-\frac{2}{3}$	$-\frac{2}{3}$

In general, if we have a representation T_a of the generators of a Lie algebra, so that $[T_a, T_b] = i{c_{ab}}^d T_d$, with real structure constants ${c_{ab}}^d$, then
$T'_a := -T^*_a$ satisfies $[T'_a, T'_b] = [T_a, T_b]^* = -i{c_{ab}}^d T^*_d = i{c_{ab}}^d T'_d$,
so T' is also a representation, called the conjugate represention to T. The weights are the eigenvalues of $T(H_i)$, which are real, so $\vec{\mu}\,' = -\vec{\mu}$. Thus the conjugate representation has a weight diagram which is just a parity reversed ($i.e.$ $\vec{\mu} \to -\vec{\mu}$) image of the original representation. The highest weight of the conjugate representation is minus the lowest weight of the original.

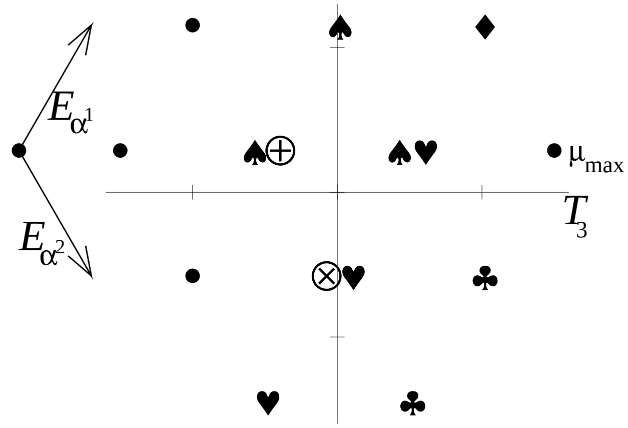
[2]To be distinguished from *weak* hypercharge.

The lowest weight of the defining representation for SU(3) is the d, and the $\bar{d} = |\mu^2\rangle$ is the highest weight of the antiquark representation. The adjoint representation of SU(3) is self-conjugate, that is, it is the same (equivalent, isomorphic) representation as its conjugate.



The adjoint representation of SU(3).

For another example, $q^1 = 2$, $q^2 = 1$, $\vec{\mu}_{\rm max} = 2\vec{\mu}^{\,1} + \vec{\mu}^{\,2} = \left(3/2, 1/2\sqrt{3}\right)$. Then to $\mu_{\rm max}$ we can apply $E_{-\alpha^1}$ twice to get 2 roots shown as ♣, and $E_{-\alpha^2}$ once to get ♢.



Now consider E_{α^1} on ♢ $= \left|1, 2/\sqrt{3}\right\rangle$. p must be zero or we would get a root vector $\left|3/2, 7/2\sqrt{3}\right\rangle$ which is higher weight than $\mu_{\rm max}$.

But $2\dfrac{\vec{\mu}\cdot\vec{\alpha}^1}{(\alpha^1)^2} = q - p = 2\left(1, \dfrac{2}{\sqrt{3}}\right)\cdot\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right) = 3$, so $E_{-\alpha^1}$ on ♢ generates three states, shown with ♡.

Next consider the higher ♣ $= \left|1, -1/\sqrt{3}\right\rangle$, acted on by E_{α^2} The p is zero because

$$E_{\alpha^2}\left|1, -1/\sqrt{3}\right\rangle \propto E_{\alpha^2}E_{-\alpha^1}\left|\mu_{\rm max}\right\rangle = E_{-\alpha^1}\underbrace{E_{\alpha^2}\left|\mu_{\rm max}\right\rangle}_{=0}$$

where the first = is because different simple raising and lowering operators commute and the = 0 is because you can't raise the highest weight. So $q = 2\dfrac{\left(1, -1/\sqrt{3}\right)\cdot\left(\frac{1}{2}, -\sqrt{3}/2\right)}{1} = 2$, Thus $E_{-\alpha^2}$ on $\left|1, -1/\sqrt{3}\right\rangle$ generates the two ♠ states.

Now the question arises whether the ♡ and ♠ at $\left(\dfrac{1}{2}, \dfrac{1}{2\sqrt{3}}\right)$ are the same state or not. This amounts to asking whether

$$E_{-\alpha^2}E_{-\alpha^1}\,|\mu_{\rm max}\rangle \quad \text{and} \quad E_{-\alpha^1}E_{-\alpha^2}\,|\mu_{\rm max}\rangle$$

are linearly independent. You will show (problem 9.A) that they are linearly independent, so there are in fact two states corresponding to the weight $\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right)$.

Rather than continuing down and to the left, we will find that the symmetries of the root diagram will determine the rest.

Given any root, there is a reflection which one can perform analogous to $e^{i\pi L_2}$ which reflects the weight vector of any representation. To see this, consider, for any root $\vec{\alpha}$, not necessarily simple,

$$[E_\alpha - E_{-\alpha}, H_i] = -\alpha_i\left(E_\alpha + E_{-\alpha}\right).$$

We now consider two cases. If $\vec{\beta}\cdot\vec{\alpha}=0$,

$$\left[E_\alpha - E_{-\alpha}, \vec{\beta}\cdot\vec{H}\right] = 0,$$

while

$$\left[E_\alpha - E_{-\alpha}, \vec{\alpha}\cdot\vec{H}\right] = -\alpha^2\left(E_\alpha + E_{-\alpha}\right).$$

Suppose we consider the state

$$|\psi\rangle = e^{-t\left(E_\alpha - E_{-\alpha}\right)}\,|\vec{\mu},D\rangle$$

where $\vec{\mu}$ is an arbitrary basis state with weight vector μ_i in an arbitrary representation D^i. The Cartan generators

$$\begin{aligned}\vec{\beta}\cdot\vec{H}\,|\psi\rangle &= \vec{\beta}\cdot\vec{H}e^{-t\left(E_\alpha - E_{-\alpha}\right)}\,|\vec{\mu},D\rangle\\ &= e^{-t\left(E_\alpha - E_{-\alpha}\right)}\vec{\beta}\cdot\vec{H}\,|\vec{\mu},D\rangle = \vec{\beta}\cdot\vec{\mu}\,|\psi\rangle\end{aligned}$$

for $\vec{\beta}\cdot\vec{\alpha}=0$.

To calculate $\vec{\alpha}\cdot\vec{H}\,|\psi\rangle = \vec{\alpha}\cdot\vec{H}e^{-t\left(E_\alpha - E_{-\alpha}\right)}\,|\vec{\mu},D\rangle$ we use the general expression

$$e^{tA}Be^{-tA} = \sum_{n=0}^\infty \frac{t^n}{n!}\Omega_n(A,B),$$

where Ω_n means the n'th multiple commutator:

$$\Omega_0(A,B) = B, \qquad \Omega_n(A,B) = [A, \Omega_{n-1}(A,B)] \,.$$

Let $A = E_\alpha - E_{-\alpha}$, $B = \vec{\alpha} \cdot \vec{H}$,

$$\begin{aligned}
\Omega_1 &= -\alpha^2 \left(E_\alpha + E_{-\alpha}\right), \\
\Omega_2 &= -\alpha^2 \left[E_\alpha - E_{-\alpha}, E_\alpha + E_{-\alpha}\right] \\
&= -2\alpha^2 \left[E_\alpha, E_{-\alpha}\right] = -2\alpha^2 \vec{\alpha} \cdot \vec{H}.
\end{aligned}$$

$$\text{Thus} \quad \Omega_n = \begin{cases} (-2\alpha^2)^{n/2} \vec{\alpha} \cdot \vec{H} & n \text{ even} \\ (-2\alpha^2)^{\frac{n-1}{2}} (-\alpha^2) \left(E_\alpha + E_{-\alpha}\right) & n \text{ odd} \end{cases},$$

and $e^{tA} \vec{\alpha} \cdot \vec{H} e^{-tA} = \vec{\alpha} \cdot \vec{H} \cos(t\sqrt{2\alpha^2}) - \sqrt{\dfrac{\alpha^2}{2}} \left(E_\alpha + E_{-\alpha}\right) \sin(t\sqrt{2\alpha^2})$. Let $t = \frac{\pi}{\sqrt{2\alpha^2}}$, so $e^{tA} \vec{\alpha} \cdot \vec{H} e^{-tA} = -\vec{\alpha} \cdot \vec{H}$. Now $\vec{\alpha} \cdot \vec{H} \left|\psi\right\rangle = \vec{\alpha} \cdot \vec{H} e^{-tA} \left|\vec{\mu}, D\right\rangle = e^{-tA} (-\vec{\alpha} \cdot \vec{H}) \left|\vec{\mu}, D\right\rangle = -\vec{\alpha} \cdot \vec{\mu} \left|\psi\right\rangle$. But for $\vec{\beta} \cdot \vec{\alpha} = 0$, $\vec{\beta} \cdot \vec{H} \left|\psi\right\rangle = \vec{\beta} \cdot \vec{\mu} \left|\psi\right\rangle$.

Now any vector $\vec{\gamma}$ can be written

$$\vec{\gamma} = \frac{\vec{\gamma} \cdot \vec{\alpha}}{\alpha^2} \vec{\alpha} + \vec{\beta}, \qquad \vec{\beta} = \vec{\gamma} - \frac{\vec{\gamma} \cdot \vec{\alpha}}{\alpha^2} \vec{\alpha}, \qquad \vec{\beta} \cdot \vec{\alpha} = 0.$$

$$\begin{aligned}
\vec{\gamma} \cdot \vec{H} \left|\psi\right\rangle &= \left(\vec{\beta} - \frac{\vec{\gamma} \cdot \vec{\alpha}}{\alpha^2} \vec{\alpha}\right) \cdot \vec{\mu} \left|\psi\right\rangle \\
&= \left(\vec{\gamma} - 2\frac{\vec{\gamma} \cdot \vec{\alpha}}{\alpha^2} \vec{\alpha}\right) \cdot \vec{\mu} \left|\psi\right\rangle .
\end{aligned}$$

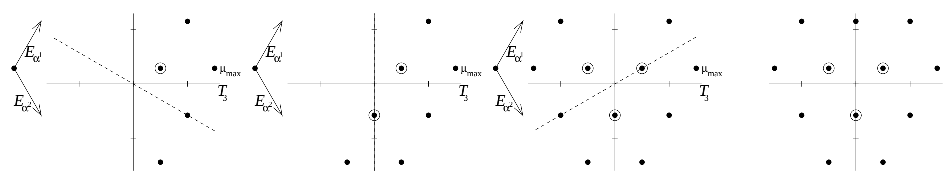
Each generator H_i corresponds to $\gamma_j = \delta_{ij}$,

$$H_i \left|\psi\right\rangle = \mu_i - 2 \,\frac{\vec{\mu} \cdot \vec{\alpha}}{\alpha^2} \alpha_i \left|\psi\right\rangle,$$

so ψ has weight vector $\vec{\mu}\,' = \vec{\mu} - 2 \,\dfrac{\vec{\mu} \cdot \vec{\alpha}}{\alpha^2} \,\vec{\alpha}$.

The transformation $e^{-\dfrac{\pi}{\sqrt{2\alpha^2}} \left(E_\alpha - E_{-\alpha}\right)}$ is a unitary transformation, so it makes a 1-1 correspondence between weights of weight $\vec{\mu}$ and those of weight $\vec{\mu} - \frac{2\,\vec{\alpha} \cdot \vec{\mu}}{\alpha^2} \,\vec{\alpha}$. This corresponds to reflection in a plane (or hyperplane) perpendicular to $\vec{\alpha}$. Thus the weight diagram of any representation must be symmetric under such reflections.

From the part of the $(2,1)$ representation we have found so far, as shown in (a), we can reflect in the plane (line) perpendicular to α^1, to get the states in (b). Then we reflect perpendicular to T_3 to get (c), then perpendicular to α^2 to get the full representation, or multiplet (d).



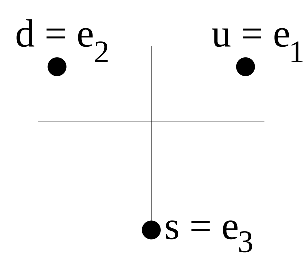
(a) (b) (c) (d)

This is the **<u>15</u>** representation of SU(3).

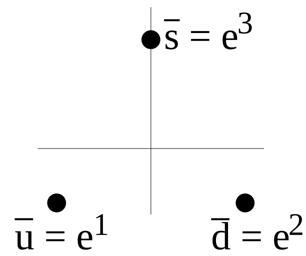
The reflection about the hyperplane perpendicular to $\vec{\alpha}^i$ is known as a *Weyl reflection*, and the group generated by all such reflections for a given algebra is called the *Weyl group.* Any representation must be invariant under the Weyl group.

10.2 Tensor Methods

Consider the basis vectors e_1, e_2, e_3 of the defining representation of SU(3). The algebra generators act by $T_a e_i = e_j \left(T_a\right)^j{}_i$ with $\left(T_a\right)^j{}_i = \frac{1}{2}\left(\lambda_a\right)_{ji}$ (and we are using the summation convention that an index appearing once upstairs and once downstairs in a term is understood to be summed over).



From this point on we must be careful with upper and lower indices in another sense than in our previous discussion. Here we are using them in the sense of co- and contra-variant quantities, as is done in relativity.



We consider the basis vectors of the conjugate representation $\bar{u} = e^1$, $\bar{d} = e^2$, $\bar{s} = e^3$. The generators act here with the conjugate representation $T' = -T^*$, so

$$T_a e^i = e^j \left(T'_a\right)_j{}^i = -e^j T^*{}_j{}^i = -e^j T^i{}_j,$$

as λ is hermitean.

Now if we consider a tensor product, $e^{ij}{}_k = e^i \otimes e^j \otimes e_k$, the Lie algebra generators act as a sum of pieces (like a derivative does), so

$$T_a e^{ij}{}_k = e^{ij}{}_m T_a{}^m{}_k - e^{mj}{}_k T_a{}^i{}_m - e^{im}{}_k T_a{}^j{}_m.$$

A vector $\vec{v}$ can be specified in terms of its components, which we give indices to so as to contract with the basis vectors. Thus a vector in the defining representation is

$$|v\rangle = v^i e_i, \quad T_a |v\rangle = v^i e_j T_a{}^j{}_i =: |\delta v\rangle = \delta v^j e_j,$$

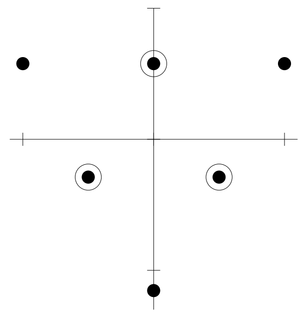
with $\delta v^i = T_a{}^i{}_j v^j$.

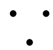
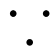
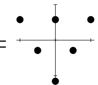
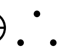
A vector in the tensor product space has coefficients with several indices

$$\begin{aligned} |v\rangle &= v_{ij}{}^k e^{ij}{}_k, \qquad T_a |v\rangle = |\delta v\rangle = \delta v_{ij}{}^k e^{ij}{}_k \\ \text{where} \quad \delta v_{ij}{}^k &= T^k{}_m v_{ij}{}^m - T^m{}_i v_{mj}{}^k - T^m{}_j v_{im}{}^k. \end{aligned}$$

The set of all states $\vec{v} = v_{ij}{}^k e^{ij}{}_k$, for arbitrary v, clearly form a representation, but it is also clearly not irreducible, because the operation of the group does nothing that alters the symmetry of the v's under $i \leftrightarrow j$). That is, suppose we start off with a particular state $v_{ij}{}^k$, and divide it into parts $v_{ij}{}^k = s_{ij}{}^k + a_{ij}{}^k$ symmetric and antisymmetric under $i \leftrightarrow j$). Then $s_{ij}{}^k$ is mapped into other symmetric coefficients under the group operations, while a is mapped into other antisymmetric ones, and they don't mix, so we have reduced the representation into two.

As an example, consider $\mathbf{3} \otimes \mathbf{3}$, where $\mathbf{3}$ is the usual name for the defining representation  with $q^i = (1, 0)$. The tensor product is a 9 dimensional representation with nine basis vectors $e_i \otimes e_j$ having weights $\mu(e_i) + \mu(e_j)$, with a general vector $|v\rangle = v^{ij} e_i \otimes e_j$. The three basis vectors $e_i \otimes e_j$ with $i = j$ have weights which can only be composed in one way, but the ones with $i \neq j$ have the same weight for i, j and for j, i. So the weight diagram is as shown.



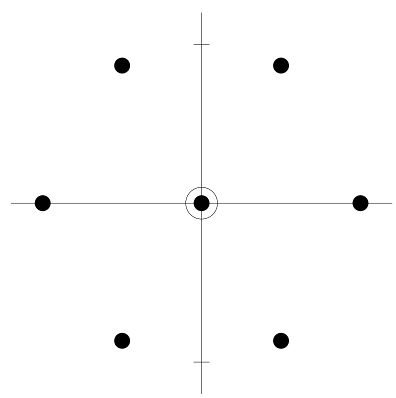
Dividing the space of 3×3 matrices v into symmetric ones s^{ij} and antisymmetric ones a^{ij}, we see that s^{ij} forms a six dimensional space and a^{ij} a three dimensional one. This divides  $\bigotimes$  $=$  $\bigoplus$  $= \mathbf{6} + \bar{\mathbf{3}}$, each of which is irreducible. Note that we can write the antiquark $\bar{\mathbf{3}}$, one of the fundamental representations, in terms of the quark, or defining, representation.

There is something else left invariant by the generators of SU(3). If we trace (*i.e.* set equal and sum) a lower with an upper index, the corresponding terms in δv cancel.

The simplest example is ${v_i}^j$ of $\bar{\mathbf{3}} \otimes \mathbf{3}$. The trace $v = \sum_i {v_i}^i$ is invariant under the generators,

$$\delta {v_i}^i = T^i_{\ m} {v_i}^m - T^m_{\ i} {v_m}^i = 0.$$

Writing ${v_i}^j = \frac{1}{3}\, w\, {\delta_i}^j + {w_i}^j$ with $w = {v_i}^i$, which ensures that ${w_i}^j$ is traceless, reduces the nine-dimensional representation $\bar{\mathbf{3}} \otimes \mathbf{3}$ into the irreducible representations $\mathbf{1}$ and $\mathbf{8}$, where $\mathbf{1}$ is the identity representation (all group elements are represented by 1, and the generators by zero), and $\mathbf{8}$, called the octet, is the adjoint representation,



The adjoint (or octet) representation.

Notice that $|v\rangle := {\delta_i}^j e^i \otimes e_j$ is actually an invariant under the group, even though it looks like a tensor, *i.e.* both of its indices do transform under the group.

If we have a representation defined by some tensor coefficients ${v_{ij}}^k$ which transform in the correct manner under the group, then the trace in any upper index paired with any lower index extracts a tensor, with fewer indices, which transforms appropriately: $v_j = \sum_i {v_{ij}}^i$ transforms properly as a $\bar{\mathbf{3}}$.

If we have two representations transforming properly, say u^{ij} and ${v_{ij}}^k$, the tensor product ${w^{ij}}_{\ell m}{}^k = u^{ij}{v_{\ell m}}^k$ is a tensor which transforms properly. It can be contracted to form reduced representations, *e.g.* $w^k = {w^{ij}}_{ij}{}^k = u^{ij}{v_{ij}}^k$ is a $\mathbf{3}$. So this is one way of extracting a smaller representation from the tensor product. But we can also impose symmetries to reduce it.

We will now construct the arbitrary SU(3) irreducible representation (n,m) from reducing a tensor product of n defining representations and m of its conjugate.

First construct $w = D^{(n,0)}$. $\vec{\mu}_{\rm max} = n\vec{\mu}^1$, so $|\vec{\mu}_{\rm max}\rangle = e_1 \otimes e_1 \otimes \cdots e_1$ (n of them). This state corresponds to the tensor $w^{j_1\cdots j_n} = \prod_i \delta_{j_i,1}$ which is clearly symmetric under interchange of any two indices. How many components of w of the full representation are there? One needs only to know how many 1's, 2's and 3's are picked to make a total of n. If you choose r 1's, there are $n-r+1$ choices of how many 2's to pick, so in total there are

$$\sum_{r=0}^{n}(n-r+1) = \sum_{1}^{n+1} j = \binom{n+2}{2} = \frac{(n+2)!}{n!\,2!} = \frac{1}{2}(n+1)(n+2)$$

choices, so w has $(n+1)(n+2)/2$ independent components, and the $D^{(n,0)}$ representation is $(n+1)(n+2)/2$ dimensional.

The same argument applies to the representation $D^{(0,m)}$ of weight $\vec{\mu}_{\rm max} = m\vec{\mu}^2$,

$$|\vec{\mu}_{\rm max}\rangle = e^2 \otimes e^2 \otimes \cdots e^2, \qquad \text{(with } m \text{ factors)}.$$

So the $u_{k_1\cdots k_m}$ corresponding to this representation is totally symmetric in all its indices, and is a $(m+1)(m+2)/2$ dimensional representation.

The tensor product of $w^{j_1\cdots j_n}$ with $u_{k_1\cdots k_m}$ can be reduced

$$w^{j_1\cdots j_n}u_{k_1\cdots k_m} = v^{j_1\cdots j_n}_{k_1\cdots k_m} + \sum_{rs} \delta^{j_r}_{k_s} X^{j_1\cdots \hat{j}_r \cdots j_n}_{k_1\cdots \hat{k}_s \cdots k_m},$$

where $\hat{j}_r$ means leave out the j_r index. The division is arranged so that v is traceless,

$$\sum_{j_1} v^{j_1 j_2\cdots j_n}_{j_1 k_2\cdots k_m} = 0.$$

The $\binom{n+2}{2}\binom{m+2}{2}$ degrees of freedom in wu have had $\binom{n-1+2}{2}\binom{m-1+2}{2}$ degrees of freedom in X constrained out, leaving

$$\begin{aligned}\text{Dim } D^{(n,m)} &= \frac{(n+2)(n+1)(m+2)(m+1)}{4} - \frac{(n+1)n(m+1)m}{4} \\ &= \frac{(n+1)(m+1)(n+m+2)}{2}\end{aligned}$$

for the dimension of the (n,m) representation.

We have extracted the leading irreducible representation from the product, but we have not fully reduced the product. Although v is irreducible, we could reduce X iteratively in the same way, extracting the traces and being left with traceless parts. Thus we find

$$D^{(n,0)} \otimes D^{(0,m)} = D^{(n,m)} \oplus D^{(n-1,m-1)} \oplus \cdots \oplus D^{(n-m,0)}$$

if $n \geq m$, or ending with $D^{(0,m-n)}$ if $m > n$.

We have constructed an arbitrary representation of SU(3) by products of **3**'s and $\bar{\mathbf{3}}$'s, but we have also seen that $\bar{\mathbf{3}}$ is the antisymmetric part in the product of two **3**'s. So any representation can be built of the defining representation alone. This is a general feature of SU(n). The states, however, will not correspond to some simple symmetry under permutations. Consider $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$, the states of three quarks. The highest weight is 3 u's, so $(3,0) = \mathbf{10}$ is the symmetric part. There is also a totally antisymmetric part w^{ijk}, but as there is only one choice for $\{i,j,k\}$ which doesn't vanish by antisymmetry, there is only one degree of freedom here, $w^{ijk} = w\epsilon^{ijk}$, so this is the one dimensional identity representation **1**. The remaining 16 degrees of freedom are in fact two octets (two **8**'s). We will have to show this.

For rank greater than 2, using $\bigotimes_i (D^i)^{q_i}$ requires 3 or more *sets* of indices, and our notational skills are not up to that. But, as we saw for SU(3), a fundamental representation may be extracted from a tensor product of copies of the defining one. In Chapter 13, Georgi shows that for SU(N), all fundamental representations can be extracted from tensor products of the *defining* representation, with mixed symmetries.

If we start with the tensor product of k defining representations of SU(n), we have an n^k dimensional space which is not only a representation of SU(n) but also of S_k, the permutation group on the indices. In fact, these operations commute, so we may reduce the space into simultaneous representations. So we must first learn something about representations of the permutation group.