Proof of $e^{tA}Be^{-tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\Omega_n(A,B)$

Define $\Omega_n(A,B)$ recursively by

$$\Omega_0(A,B) = B, \quad \Omega_n(A,B) = [A, \Omega_{n-1}(A,B)]\,.$$

Let $f(t) = e^{tA}Be^{-tA}$ and $g(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\Omega_n(A,B)$. Note that at $t=0$, $f(0) = B = g(0)$.

Taking derivatives,

$$\frac{df}{dt} = e^{tA}ABe^{-tA} + e^{tA}B(-A)e^{-tA} = \left[A, e^{tA}Be^{-tA}\right] = [A, f(t)]\,,$$

while

$$\begin{aligned}\frac{dg}{dt} &= \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}\Omega_n(A,B) = \sum_{m=0}^{\infty} \frac{t^m}{m!}\Omega_{m+1}(A,B) = \sum_{m=0}^{\infty} \frac{t^m}{m!}\left[A, \Omega_m(A,B)\right] \\ &= [A, g(t)]\,.\end{aligned}$$

Thus $f(t)$ and $g(t)$ satisfy the same first order ordinary differential equation and agree at $t=0$, so they are equal for all t, at least for which neither side blows up. If A and B are finite-dimensional matrices both sides converge for all t.