Physics 618 Homework #4

Due: Thursday, Feb. 16, 2017 at 4:00 PM

1 [11 pts.] Consider the 2×2 matrices with integer matrix elements and determinant 1. This set forms a group under ordinary matrix multiplication, called the *modular group* or $SL(2, \mathbb{Z})$.
a) [4 pts.] Show it is a group.
b) [5 pts.] Show it can be generated by the elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
c) [2 pts.] Is it a finite group? An abelian group? A Lie group?

2 [9 pts.] This problem is an extended exploration of the permutation group S_n. Parts (a–e) are for mathematical completeness only, and are not to be handed in. Please hand in only parts (f-h).
Note that any element of S_{n-1} can be viewed as an element of S_n which leaves n unmoved.
(a) Show that any element $g \in S_n$, $g = \begin{pmatrix} 1 & 2 & \ldots & n \\ r_1 & r_2 & \ldots & r_n \end{pmatrix}$ is either such an element of S_{n-1} or can be written as such an element followed by a transposition,

$$g = (n\ r_n)B, B \in S_{n-1}.$$

(b) Therefore, by induction, that any element of S_n can be written as a product of fewer than n transpositions.

(c) If we have a given permutation g, we can start with a given object and ask where that object is moved to, where the object that had been in this new location is moved to, and that one, and so on, until you find a object placed where the first one had been. For this subset of objects, g acts as a cycle C, say of length r. Explain that $C^{-1}g$ is a permutation on $n-r$ objects. Hence by induction, that any permutation can be written as (or presented as) a product of disjoint cycles (*i.e.* the different cycles have no objects in common).

(d) By presenting any permutation by such a product of disjoint cycles, show that conjugating by a transposition $(a\ b)$ gives a permutation given by the same product of cycles, except that the occurence of a and b in this presentation is interchanged.

(e) Note that if g is written as a product of disjoint cycles, g^{-1} is the product of the corresponding inverse cycles, which are also disjoint. [Note the inverse cycle is just the cycle "written backwards".] We want to prove that the presentation of a permutation g as a product of disjoint cycles is unique, except that the order of multiplication of the cycles is irrelevant (disjoint cycles commute with each other), and a given r-cycle can be written r ways [(123)=(231)=(312)]. To prove this uniqueness, assume you have two different presentations of g, invert one and multiply. Show that for the result to be one, the two cycles which contain object a must be inverses, for all a, which contradicts that the two representations were really different.

(f) Include 1-cycles for each object unmoved by the permutation. Then we have a presentation of the permutation g as a product of disjoint cycles of various lengths, say i_r cycles of length r, $i_r \geq 0$. There are n objects in all, so $\sum r i_r = n$, and $\{i_r\}$ is a **partition of** n. [Example, one partition of 5 is $2, 0, 1, 0...$, because $5 = 1 + 1 + 3$.] By the last part, each permutation is uniquely mapped into one partition of n.

[3 pts.] Show that g and $x^{-1}gx$ are mapped into the same partition. Hint: show it first for the case where x is a transposition, and then generally.

(g) [3 pts.] Conversely, if g and h are mapped into the same partition of n, their presentations as products of disjoint cycles are of the same form, differing only by permuting the object-symbols in the expression. Show that this permutation gives a conjugacy relation $g \cong h$.

(h) [3 pts.] Therefore, that the conjugacy classes of S_n are labelled by partitions of n, the number of which is called the partition function $P(n)$, which is given by the generating function

$$\sum_{n=0}^{\infty} P(n) x^n = \prod_{j=1}^{\infty} (1 - x^j)^{-1}.$$

3 [5 pts.] If $e^C = e^{\lambda A} e^{\lambda B}$, find the expression for C in terms of A, B, and their (multiple) commutators, up to and including terms of third order in the small c-number parameter λ.