Energy in Bremsstrahlung, Note on p. 179

As discussed on pages 178-179 in Peskin and Schroeder, the radiation field in the impulse approximation for electron scattering is given by the residues of the poles at $k^0 = \pm|\vec{k}|$ in

$$A^\mu(x) = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot x} \frac{-ie}{k^2} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)$$

The residues of $1/k^2 = 1/[(k^0 - |\vec{k}|)(k^0 + |\vec{k}|)$ are $\pm(1/2|\vec{k}|)$ at $k^0 = \pm|\vec{k}|$ respectively, so

$$\begin{aligned} A^\mu_{\text{rad}}(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\vec{k}|} \Bigg\{ \left(\frac{p'^\mu}{k^0E' - \vec{k}\cdot\vec{p}'} - \frac{p^\mu}{k^0E - \vec{k}\cdot\vec{p}} \right) e^{-i(k^0t - \vec{k}\cdot\vec{x})} \\ &\qquad - \left(\frac{p'^\mu}{-k^0E' - \vec{k}\cdot\vec{p}'} - \frac{p^\mu}{-k^0E - \vec{k}\cdot\vec{p}} \right) e^{-i(-k^0t - \vec{k}\cdot\vec{x})} \Bigg\}\Bigg|_{k^0 = +|\vec{k}|} \end{aligned}$$

Reversing the sign of the integration variable $\vec{k}$ in the second term, this may be written as

$$\begin{aligned} A^\mu_{\text{rad}}(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\vec{k}|} \Bigg\{ \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right) e^{-ik\cdot x} \\ &\qquad + \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right) e^{ik\cdot x} \Bigg\}\Bigg|_{k^0 = +|\vec{k}|} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\vec{k}|} \Bigg\{ \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right) e^{-ik\cdot x} + c.c. \Bigg\}\Bigg|_{k^0 = +|\vec{k}|} \end{aligned}$$

in agreement with 6.6 of Peskin and Schroeder. Thus we have

$$A^\mu_{\text{rad}}(x) = \int \frac{d^3k}{2(2\pi)^3} \mathcal{A}^\mu(\vec{k}) \left(e^{-ik\cdot x} + e^{ik\cdot x} \right), \text{ with } \mathcal{A}^\mu(\vec{k}) = \frac{-e}{|\vec{k}|} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right).$$

Note $k_\mu \mathcal{A}^\mu(\vec{k}) = 0$ for $k^0 = |\vec{k}|$. If we define $\bar{k}^\mu := (-|\vec{k}|, \vec{k})$ (so $\bar{k}_\mu = -\bar{k}^\mu$), we also have $\bar{k}_\mu \mathcal{A}^\mu(-\vec{k}) = 0$. This will prove useful in evaluating the energy in the radiation field.

From homework #2 we learned that the energy density is the $(0,0)$ component of the Noether current associated with translations. This approach has a complication for the Maxwell field which is discussed by Jackson, who explains that a correction is necessary to get a symmetric stress-energy tensor

$$T^{\mu\nu} = F^{\mu}{}_{\rho} F^{\rho\nu} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.$$

Evaluating the energy density

$$\mathcal{H} = T^{00} = \frac{1}{4}\left(F^{\rho\sigma}F_{\rho\sigma} - 4F^{0\rho}F_{0\rho}\right) = \frac{1}{2}\left(-F^{0j}F_{0j} + F^{ij}F_{ij}\right) = \frac{1}{2}\left(E^2 + B^2)\right)$$

Note that we can write $\mathcal{H}$ in the strange fashion

$$\mathcal{H} = -\frac{1}{4}\sum_{\rho} F^{\rho\mu} F^{\rho}{}_{\mu},$$

where I have explicitly given the $\sum_\rho$, because we don't have a summation convention on two upper indices, but the $\sum_\mu$ is left implicit.

From the expression for $A^{\mu}_{\text{rad}}(x)$ we have

$$F^{\mu\nu}_{\text{rad}}(x) = -i \int \frac{d^3k}{2(2\pi)^3} \left(k^\mu \mathcal{A}^\nu(\vec{k}) - k^\nu \mathcal{A}^\mu(\vec{k})\right)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right)\Big|_{k^0=|\vec{k}|}.$$

so

$$\begin{aligned}
H &= -\frac{1}{4}\int d^3x \sum_\rho F^{\rho\mu}_{\text{rad}} F_{\text{rad}}{}^{\rho}{}_{\mu} \\
&= \frac{1}{4}\int d^3x \int \frac{d^3k}{2(2\pi)^3} \int \frac{d^3k'}{2(2\pi)^3} \\
&\qquad \sum_\rho \left(k^\rho \mathcal{A}^\mu(\vec{k}) - k^\mu \mathcal{A}^\rho(\vec{k})\right)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right) \\
&\qquad \times\left(k'^{\rho} \mathcal{A}_\mu(\vec{k}') - k'^{\mu} \mathcal{A}_\rho(\vec{k}')\right)\left(e^{-ik'\cdot x} - e^{ik'\cdot x}\right)\Big|_{\substack{k^0=|\vec{k}| \\ k'^0=|\vec{k}'|}}
\end{aligned}$$

We can use $\int d^3x d^3k' e^{i(\vec{k}\pm\vec{k}')\cdot\vec{x}} = (2\pi)^3$ with $\mp\vec{k}$ substituted for $\vec{k}'$ in the rest of the expression. As all k^0 and k'^0 are positive, it would be better to say $k' \to -\bar{k}$ in the first case.

Thus we have

$$\begin{aligned} H &= \frac{1}{4}\int \frac{d^3k}{4(2\pi)^3}\sum_\rho \Big\{ -2\left(k^\rho \mathcal{A}^\mu(\vec{k}) - k^\mu \mathcal{A}^\rho(\vec{k})\right)\left(k^\rho \mathcal{A}_\mu(\vec{k}) - k^\mu \mathcal{A}_\rho(\vec{k})\right) \\ &\quad + \left(k^\rho \mathcal{A}^\mu(\vec{k}) - k^\mu \mathcal{A}^\rho(\vec{k})\right)\left(\bar{k}^\rho \mathcal{A}_\mu(-\vec{k}) - \bar{k}^\mu \mathcal{A}_\rho(-\vec{k})\right) \\ &\qquad \times \left(e^{2i|\vec{k}|t} + e^{-2i|\vec{k}|t}\right)\Big\}\Big|_{k^0=|\vec{k}|}. \end{aligned}$$

where $\bar{k} = (-k^0, \vec{k})$. Note $\sum_\rho k^\rho V^\rho = -\bar{k}\cdot V$, and $\sum_\rho k^\rho k^\rho = 2|\vec{k}|^2$, so

$$\begin{aligned} H &= \frac{1}{16}\int \frac{d^3k}{(2\pi)^3}\Big\{ -8|\vec{k}|^2 \mathcal{A}^2(\vec{k}) + 4k\cdot\mathcal{A}(\vec{k})\bar{k}\cdot\mathcal{A}(\vec{k}) \\ &\quad + \left(-2k^2 \mathcal{A}(\vec{k})\cdot\mathcal{A}(-\vec{k}) + 2k\cdot\mathcal{A}(-\vec{k})k\cdot\mathcal{A}(\vec{k})\right)\left(e^{2i|\vec{k}|t} + e^{-2i|\vec{k}|t}\right) \\ &= -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|\vec{k}|^2 \mathcal{A}^2(\vec{k}) \end{aligned}$$

where we have used $k^2 = 0$ and $k\cdot\mathcal{A}(\vec{k}) = 0$. Thus we have

$$\begin{aligned} H &= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|\vec{k}|^2 \left(\frac{-e}{|\vec{k}|}\right)^2 \left(\frac{2p\cdot p'}{k\cdot p'\, k\cdot p} - \frac{m^2}{(k\cdot p')^2} - \frac{m^2}{(k\cdot p)^2}\right) \\ &= \frac{e^2}{2}\int \frac{d^3k}{(2\pi)^3}\left(\frac{2p\cdot p'}{k\cdot p'\, k\cdot p} - \frac{m^2}{(k\cdot p')^2} - \frac{m^2}{(k\cdot p)^2}\right) \end{aligned}$$

in agreement with 6.13.

On getting 6.17:
The last two of the three terms in the integral 6.15 for $\mathcal{I}(\vec{v},\vec{v}\,')$ can be done by aligning the spherical coordinates along the relevant velocity. The first term can also be done that way if we first combine the denominators using the Feynman parameter trick,

$$\frac{1}{AB} = \int_0^1 \frac{1}{[xA + (1-x)B]^2}dx,$$

which gives

$$\int \frac{d\Omega_k}{4\pi}\frac{1}{(\hat{k}\cdot p')(\hat{k}\cdot p)} = \int_0^1 d\alpha \frac{1}{m^2 - \alpha(1-\alpha)q^2}.$$

This is worked out in Lecture 21 notes.