Comment on Integrals over Grassman Variables

Joel A. Shapiro

With the rule for integrating over a Grassman variable,

$$\int d\theta f(\theta) = B, \qquad \text{where } f(\theta) = A + \theta B,$$

we can define nested integrals, so that if f is a function of n θ's, we may define

$$\int d^n\theta f(\{\theta_i\}) = \int d\theta_n d\theta_{n-1} \ldots d\theta_1 f(\{\theta_i\}) = \text{coefficient of } \theta_1\theta_2 \ldots \theta_n.$$

Note the order of the $d\theta$'s in $d^n\theta$.

Note that an equivalent rule for how $\int d\theta$ acts on $f(\theta)$ is to define it as: *take the θ derivative*! Thus it would appear that

$$\int d\theta_n d\theta_{n-1} \ldots d\theta_1 = \frac{\partial}{\partial\theta_n}\frac{\partial}{\partial\theta_{n-1}} \cdots \frac{\partial}{\partial\theta_1},$$

and in fact people often think about the integral as really being a derivative. Some caution is needed, however.

We will assume for the moment that our θ's are real. As we are allowed to multiply them by real numbers and add them, as a vector space, we might consider a change in variables,

$$\theta'_i = A_{ij}\theta_j,$$

where A is a real matrix. If we reexpress f in terms of those variables,

$$f(\{\theta_i\}) = f'(\{\theta'_i\}),$$

the coefficients c_n for f and c'_n for f' are related by

$$c'_n(A_{1,j_1}\theta_{j_1})(A_{2,j_2}\theta_{j_2}) \ldots (A_{n,j_n}\theta_{j_n}) = c_n\theta_1\theta_2 \ldots \theta_n.$$

Rearrangeing the θ's on the left gives $\epsilon_{j_1,j_2 \ldots,j_n}$. so

$$c_n = c'_n\epsilon_{j_1,j_2\ldots,j_n}\prod_{i=1}^{n} A_{i,j_i} = c'_n \det A.$$

If we assume the infinitesimal d (the exterior derivative) is linear over the ordinary numbers, so $d\theta'_i = A_{ij} d\theta_j$, then $d^n\theta' = \det A\, d^n\theta$ exactly as for the product of n θs, and we find

$$\int d^n\theta' f'(\{\theta'\}) = \det A \int d^n\theta f(\{\theta\}) = c_n \det A = c'_n (\det A)^2,$$

which is not what we would have found by assuming the θ''s were as valid as the θ's, unless $(\det A) = \pm 1$, as would be the case for an orthogonal matrix. Of course restricting to orthogonal transformations is reasonable.

But to define complex θ's Peskin and Schroeder define

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}, \qquad \theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}}, \tag{1}$$

which is of the form above with a complex unitary matrix A which is not orthogonal and has determinant $-i$. Thus the rest of the steps leading to their 9.66 are dubious.

If we suppose linearity of d, (1) implies

$$d\theta = \frac{d\theta_1 + i d\theta_2}{\sqrt{2}}, \qquad d\theta^* = \frac{d\theta_1 - i d\theta_2}{\sqrt{2}}, \tag{2}$$

but then

$$\int d\theta\, \theta = \int \frac{d\theta_1 + i d\theta_2}{\sqrt{2}} \frac{\theta_1 + i\theta_2}{\sqrt{2}} = \frac{1}{2} \int d\theta_1\, \theta_1 - d\theta_2\, \theta_2 = 0,$$

while

$$\int d\theta\, \theta^* = \int \frac{d\theta_1 + i d\theta_2}{\sqrt{2}} \frac{\theta_1 - i\theta_2}{\sqrt{2}} = \frac{1}{2} \int d\theta_1\, \theta_1 + d\theta_2\, \theta_2 = 1.$$

and also $\int d\theta^*\, \theta = 1$. That seems somewhat strange but might be suitable (θ and θ^* are complex so as fields would be considered charged, with opposite charge, while the integral has lost any Grassman charge and should be neutral). But it would also make

$$\int d\theta^* d\theta\, \theta\theta^* = -\int d\theta^* d\theta\, \theta^*\theta = -\int d\theta^* \left(\int d\theta\theta^* \right) \theta = -\int d\theta^*\theta = -1,$$

in disagreement with P&S.