On Evaluating $\mathcal{I}(\vec{v}, \vec{v}\,')$

The expression for

$$\mathcal{I}(\vec{v}, \vec{v}\,') = \int \frac{d\Omega_k}{4\pi} \left(\frac{2p \cdot p'}{(\hat{k} \cdot p')(\hat{k} \cdot p)} - \frac{m^2}{(\hat{k} \cdot p')^2} - \frac{m^2}{(\hat{k} \cdot p)^2} \right)$$

can be evaluated using the Feynman parameter trick. First of all the last two terms in (6.15) can be evaluted, for each choosing the z access along the velocity, so together they contribute

$$\begin{aligned} \int \frac{d\Omega_k}{4\pi} \frac{-2m^2}{E^2} \frac{1}{(1 - v\cos\theta)^2} &= \frac{-m^2}{E^2} \int_{-1}^{1} \frac{du}{(1 - vu)^2} = -\frac{m^2}{E^2 v} \frac{1}{1 - vu}\Big|_{-1}^{1} \\ &= -\frac{m^2}{E^2 v} \left(\frac{1}{1 - v} - \frac{1}{1 + v} \right) = -\frac{2m^2}{E^2} \frac{1}{1 - v^2} = -2. \end{aligned}$$

For the first term, use the Feynman trick

$$\frac{1}{(\hat{k} \cdot p')(\hat{k} \cdot p)} = \int_0^1 d\alpha \frac{1}{(\hat{k} \cdot (\alpha p' + (1 - \alpha)p))^2}.$$

Recalling we are working in a frame with $E' = E$ and $q^0 = 0$, this is just like $1/2m^2$ times the above with $Ev \to \alpha(\vec{p}\,' - \vec{p}) + \vec{p} = \alpha\vec{q} + \vec{p}$, so the integral is $\dfrac{1}{E^2 - \vec{p}^{\,2} - 2\alpha\vec{p} \cdot \vec{q} - \alpha^2\vec{q}^{\,2}}$. As $\vec{p} \cdot \vec{q} = \vec{p} \cdot \vec{p}\,' - \vec{p}^{\,2} = -\frac{1}{2}\vec{q}^{\,2} = q^2/2$, we have

$$\int \frac{d\Omega_k}{4\pi} \frac{1}{(\hat{k} \cdot p')(\hat{k} \cdot p)} = \int_0^1 d\alpha \frac{1}{m^2 - \alpha(1 - \alpha)q^2}.$$

$2p \cdot p' = 2m^2 - q^2$ so all together,

$$\mathcal{I}(\vec{v}, \vec{v}\,') = \int_0^1 \left(\frac{2m^2 - q^2}{m^2 - \alpha(1 - \alpha)q^2} \right) d\alpha - 2 =: 2f_{\mathrm{IR}}(q^2).$$

If $-q^2 \gg m^2$, the integral is given by equal contributions near each endpoint, so $\approx 2\displaystyle\int_0 d\alpha \frac{1}{\alpha - m^2/q^2} \approx 2\ln(-q^2/m^2)$.