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Non-Abelian Gauge Theory

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Today we are going to begin a discussion of Gauge Field Theory, not just for a better understanding of the Maxwell Field of QED, but also to generalize gauge invariance to non-Abelian groups. Non-Abelian gauge fields are the basis not only of Quantum Chromodynamics (QCD), our current understanding of the strong interactions, but also the foundation of the Glashow-Salam-Weinberg theory of electroweak interactions.

I am going to be somewhat more explicit than Peskin and Schroeder, in using latticization to understand the meaning of the gauge fields and the covariant derivatives.

Gauge fields are all about continuous symmetry groups (Lie groups) and I don't know how much you already know about these. I will assume you at least know the rotation group and its Lie algebra and arbitrary spin representations, as we have already used this in analyzing the representations of the Poincaré group. The general discussion for a wider set of groups does not introduce much beyond what you can already see with SO(3), but there are some web pages on group theory which can fill in your knowledge

- "Lightning review of groups".
- "Notes on Representations, Adjoint Rep., Killing Form, and the Antisymmetry of ${c_{ij}}^k$".
- "Group Invariant Metric".

We begin with a theory that has a non-Abelian internal symmetry. That is, it is invariant under G: $\phi'_i(x) = M_{ij}(g)\phi_j(x)$, for $g \in G$, where M is a linear representation of the group G, and g is a fixed group element, the same at every space-time point. This is called Global Symmetry. Our standard theory, say $\mathcal{H}_\text{I} = \left(\sum_1^N \phi_j^2\right)^2$, is invariant under $SO(N)$. But it is not invariant if we perform $SO(N)$ transformations which vary from one space-time point to another. That is, it does not have *local* SO(N) symmetry. We are going to ask how we can deform our theory so that it **is invariant** under this much larger set of symmetry transformations,

$$\phi'_i(x) = M_{ij}(g(x))\phi_j(x).$$

For $SO(N)$ we can think of the group action as a change of coordinate systems for expression the N-vector $\vec{\phi}$, so we are looking for a theory which is invariant under independently changing coordinate systems, or basis vectors, at each point. Terms without derivatives in the lagrangian are automatically locally invariant if they are globally, but terms involving derivatives, which compare fields at different points, require a definition of parallel transport. In ordinary (non-gauge) theories, this definition is fixed, and we usually use basis vectors for which parallel transport means leaving the components unchanged. But in Gauge theories, the law of parallel transport becomes dynamical. At each point $\mathbf{x}$ and each direction μ, we have an infinitesimal generator of the group, or Lie algebra element,

$$\mathcal{A}_\mu(x) = A_\mu^{(b)} L_b.$$

We could then make a **locally** gauge invariant theory by replacing the derivative by the **covariant derivative**

$$\partial_\mu \phi_i \to (D_\mu \phi)_i = \partial_\mu \phi_i - igM_{ij}(\mathcal{A}_\mu)\phi_j = \partial_\mu \phi_i - igA_\mu^{(b)} M_{ij}(L_b)\phi_j.$$

Then

$$\begin{aligned} \phi'_i(x) &= \left(e^{i\lambda^{(b)}(x)M(L_b)}\right)_{ij} \phi_j(x) \\ \mathcal{A}'_\mu &= e^{i\lambda}\left(\mathcal{A}_\mu + \frac{i}{g}\partial_\mu\right)e^{-i\lambda} \qquad \left(\lambda = \lambda^{(b)} L_b\right) \end{aligned}$$

implements the local symmetric of the theory, with $G(x) = e^{i\lambda}$.

This is just an introduction to the main part of the lecture, in which this is explained in terms of lattice theory in “Gauge Theory on a Lattice”.