Lecture 21 Nov. 14, 2013

$\delta F_1(q^2)$.

Last time we saw that the calculation of the first order (in α) correction to F_2 was untroubled by infrared or ultraviolet divergences, but the expression for the first order correction to F_1,

$$\begin{aligned}\delta F_1(q^2) &= 2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int dx\,dy\,dz\,\delta(1-x-y-z)\\ &\qquad\frac{-\ell^2+2(1-x)(1-y)q^2+2(1-4z+z^2)m^2}{\left(\ell^2-\Delta+i\epsilon\right)^3},\end{aligned}$$

(with $\Delta=-xy\,q^2+(1-z)^2m^2$), diverges in the ultraviolet because of the term ℓ^2 in the numerator, and also, at $q=0$, because $\Delta=(1-z)^2m^2$ vanishes in the denominator at the $z\approx 1$ end of the integration interval.

We may regulate the infrared divergence by pretending that the photon has a small mass μ instead of being massless, thereby changing the photon propagator's denominator $(k-p)^2+i\epsilon\to(k-p)^2-\mu^2+i\epsilon$, which changes $\Delta\to -xy\,q^2+(1-z)^2m^2+z\mu^2$. To take care of the ultraviolet divergence, pretend that there is also another, very heavy, photon of mass Λ with imaginary coupling, so that there is another term, and now the photon propagator

$$\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\to\frac{-ig_{\nu\rho}}{(k-p)^2-\mu^2+i\epsilon}-\frac{-ig_{\nu\rho}}{(k-p)^2-\Lambda^2+i\epsilon},$$

Eventually we will take $\mu\to 0$ and $\Lambda\to\infty$, and in terms without ultraviolet divergences the heavy photon's contribution will vanish. Using the second and last expressions from page 3 of last time's notes, this gives

$$\begin{aligned}\delta F_1(q^2) &= \frac{2}{4\pi}\frac{e^2}{4\pi}\int dx\,dy\,dz\,\delta(1-x-y-z)\\ &\qquad\left[\ln\frac{-xy\,q^2-(1-z)^2m^2+z\,\Lambda^2}{-xy\,q^2+(1-z)^2m^2+z\,\mu^2}\right.\\ &\qquad\qquad\left.+\frac{(1-x)(1-y)q^2+(1-4z+z^2)m^2}{-xy\,q^2+(1-z)^2m^2+z\,\mu^2}\right].\end{aligned}$$

As we are interested in the $\Lambda \to \infty$ limit, we can drop the other terms in the numerator of the log. For $q^2 = 0$ the integrand is independent of x and y so $\int dx\, dy\, dz\, \delta(1-x-y-z) \to \int_0^1 dz(1-z)$, and

$$\delta F_1(0) = \frac{\alpha}{2\pi}\int_0^1 dz\,(1-z)\left[\ln\frac{z\,\Lambda^2}{(1-z)^2m^2+z\,\mu^2}+\frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\,\mu^2}\right],$$

which is 7.32, and what we will need in explaining how to throw $\delta F_1(0)$ away.

But first we will ask about what is left of $F_1(q^2)$ after we throw away the troublesome pieces we know ought not make $F_1(0)$ differ from 1. That is, define

$$\begin{aligned}\bar{\delta}F_1(q^2) &:= \lim_{\Lambda\to\infty}\Big(F_1(q^2)-F_1(0)\Big)\\ &= \frac{\alpha}{2\pi}\int dx\,dy\,dz\,\delta(1-x-y-z)\\ &\Bigg[\ln\frac{(1-z)^2m^2+z\mu^2}{-xy\,q^2+(1-z)^2m^2+z\mu^2}\\ &\quad+\frac{(1-x)(1-y)q^2+(1-4z+z^2)m^2}{-xy\,q^2+(1-z)^2m^2+z\,\mu^2}-\frac{(1-4z+z^2)m^2}{(1-z)^2m^2+z\,\mu^2}\Bigg].\end{aligned}$$

The logarithm is not singular as $\mu \to 0$ so we can drop those terms, and this term gives $\displaystyle\frac{\alpha}{2\pi}\int_0^1(1-z)dz\int_0^1 d\xi\ln\frac{m^2}{-\xi(1-\xi)\,q^2+m^2}$, which is nonsingular, where we substituted $x = (1-z)\xi$.

Now the part of $\bar{\delta}F_1(q^2)$ which does blow up for $\mu^2 \to 0$ comes from the $z = 1, x = y = 0$ endpoint of the integral, so except for vanishing denominators, we can make that substitution, and the same substitution as above, $x = (1-z)\xi$, and also $w = 1-z$, to get

$$\begin{aligned}\bar{\delta}F_1(q^2) &\sim \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z}dx\frac{q^2-2m^2}{m^2(1{-}z)^2{-}q^2x(1{-}z{-}x){+}\mu^2}-\frac{-2m^2}{m^2(1{-}z)^2{+}\mu^2}\\ &= \frac{\alpha}{4\pi}\int_0^1 d(w^2)\int_0^1 d\xi\frac{q^2-2m^2}{(m^2-q^2\xi(1-\xi))w^2+\mu^2}-\frac{-2m^2}{m^2w^2+\mu^2}\\ &= \frac{\alpha}{4\pi}\int_0^1 d\xi\frac{q^2-2m^2}{(m^2{-}q^2\xi(1{-}\xi))}\ln\left(\frac{(m^2{-}q^2\xi(1{-}\xi){+}\mu^2}{\mu^2}\right)+\frac{\alpha}{2\pi}\ln\left(\frac{m^2}{\mu^2}\right).\end{aligned}$$

Thus

$$F_1(q^2) = 1-\frac{\alpha}{2\pi}f_{\rm IR}(q^2)\ln\left(\frac{m^2}{\mu^2}\right)+\text{IR nonsingular terms},$$

where

$$f_{\rm IR}(q^2) = \int_0^1 \left(\frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)} \right) d\xi - 1.$$

Read the first third of page 200

So the next order corrections to the elastic scattering amplitude subtracts a piece proportional to the lowest order calculation. But we also saw that the lowest order calculation of the cross section for emission of a soft photon of energy less than ε was similarly proportional to the elastic scattering,

$$d\sigma(\vec{p} \to \vec{p}\,' + \gamma) = d\sigma(\vec{p} \to \vec{p}\,') \cdot \frac{\alpha}{2\pi} \ln\left(\frac{\varepsilon^2}{\mu^2}\right) \mathcal{I}(\vec{v}, \vec{v}\,'),$$

On Evaluating $\mathcal{I}(\vec{v}, \vec{v}\,')$

The expression for

$$\mathcal{I}(\vec{v}, \vec{v}\,') = \int \frac{d\Omega_k}{4\pi} \left(\frac{2p \cdot p'}{(\hat{k} \cdot p')(\hat{k} \cdot p)} - \frac{m^2}{(\hat{k} \cdot p')^2} - \frac{m^2}{(\hat{k} \cdot p)^2} \right)$$

can be evaluated using the Feynman parameter trick. First of all the last two terms in (6.15) can be evaluted, for each choosing the z access along the velocity, so they contribute

$$\begin{aligned} \int \frac{d\Omega_k}{4\pi} \frac{-2m^2}{E^2} \frac{1}{(1 - v\cos\theta)^2} &= \frac{-m^2}{E^2} \int_{-1}^1 \frac{du}{(1-vu)^2} = -\frac{m^2}{E^2 v} \left. \frac{1}{1-vu} \right|_{-1}^1 \\ &= -\frac{m^2}{E^2 v} \left(\frac{1}{1-v} - \frac{1}{1+v} \right) = -\frac{2m^2}{E^2} \frac{1}{1-v^2} = -2. \end{aligned}$$

For the first term, use

$$\frac{1}{(\hat{k} \cdot p')(\hat{k} \cdot p)} = \int_0^1 d\alpha \frac{1}{(\hat{k} \cdot (\alpha p' + (1-\alpha)p))^2}.$$

Recalling we are working in a frame with $E' = E$ and $q^0 = 0$, this is just like $1/2m^2$ times the above with $Ev \to \alpha(\vec{p}\,' - \vec{p}) + \vec{p} = \alpha\vec{q} + \vec{p}$, so the integral is $\dfrac{1}{E^2 - \vec{p}^{\,2} - 2\alpha\vec{p} \cdot \vec{q} - \alpha^2 \vec{q}^{\,2}}$. As $\vec{p} \cdot \vec{q} = \vec{p} \cdot \vec{p}\,' - \vec{p}^{\,2} = -\frac{1}{2}\vec{q}^{\,2} = q^2/2$, we have

$$\int \frac{d\Omega_k}{4\pi} \frac{1}{(\hat{k} \cdot p')(\hat{k} \cdot p)} = \int_0^1 d\alpha \frac{1}{m^2 - \alpha(1-\alpha)q^2}.$$

$2p \cdot p' = 2m^2 - q^2$ so all together,

$$\mathcal{I}(\vec{v}, \vec{v}\,') = \int_0^1 \left(\frac{2m^2 - q^2}{m^2 - \alpha(1-\alpha)q^2} \right) d\alpha - 2 =: 2 f_{\mathrm{IR}}(q^2).$$

If $-q^2 \gg m^2$, the integral is given by equal contributions near each endpoint, so $\approx 2 \int_0 d\alpha \frac{1}{\alpha - m^2/q^2} \approx 2 \ln(-q^2/m^2)$.

Read the bottom third of page 200 and the top half of 201.

It would be good to at least skim section 6.5 to get the gist of the argument to all orders for infrared behavior.