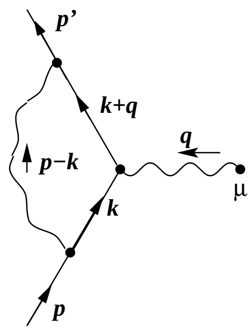
Lecture 20 Vertex Correction, $g-2$ Nov. 11, 2013






Last time we found that we could write the general form of the vertex correction in terms of

$$\Gamma^\mu(q^2) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2),$$

with $\Gamma^\mu(q^2) = \gamma^\mu + \delta\Gamma^\mu(q^2)$, and the first order correction term in $\delta\Gamma^\mu$ is given by

$$\bar{u}(p')\delta\Gamma^\mu(q^2)u(p) = 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right] u(p)}{((k-p)^2 + i\epsilon)\,(k'^2 - m^2 + i\epsilon)\,(k^2 - m^2 + i\epsilon)},$$

where $q = p' - p$, $k' = k + q$. We also saw that the three denominators can be combined using the Schwinger trick

$$\frac{1}{((k-p)^2 + i\epsilon)\,(k'^2 - m^2 + i\epsilon)\,(k^2 - m^2 + i\epsilon)} = \Gamma(3) \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z) \frac{1}{D^3},$$

where

$$\begin{aligned} D &= x\left(k^2 - m^2 + i\epsilon\right) + y\left(k'^2 - m^2 + i\epsilon\right) + z\left((k-p)^2 + i\epsilon\right) \\ &= k^2 + 2k\cdot(yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\epsilon, \end{aligned}$$

where we have substituted $k+q$ for $k'$ and used $x+y+z=1$. Recalling that $k^\mu$ is a an integration variable, we see that we can simplify the integral by shifting to $\ell = k + yq - zp$, so

$$D = \ell^2 - \Delta + i\epsilon,$$

with

$$\begin{aligned} \Delta &= (yq - zp)^2 - yq^2 - zp^2 + (x+y)m^2 \\ &= -y(1-y)q^2 - 2yz\, q\cdot p - z(1-z)p^2 + (1-z)m^2 \\ &= -y(1-y-z)q^2 + (1-z)^2 m^2 = -xyq^2 + (1-z)^2 m^2, \end{aligned}$$

where in the first expression in the third line, I used $2q\cdot p = (q+p)^2 - q^2 - p^2 = p'^2 - q^2 - p^2 = m^2 - q^2 - m^2 = -q^2$, as well as $p^2 = m^2$.

## Read pages 191–196.

Some hints:
Page 191, in the second line of the evaluation of the Numerator, the term quadratic in $\ell$ from $\not{k}\gamma^\mu \not{k}'$ is

$$\not{\ell}\gamma^\mu \not{\ell} = \ell^\alpha \ell^\beta \gamma_\alpha \gamma^\mu \gamma_\beta \sim \frac{1}{4}\ell^2 g^{\alpha\beta}\gamma_\alpha\gamma^\mu\gamma_\beta = -\frac{1}{2}\ell^2\gamma^\mu$$

by A.29(b).

Clearing the smoke on page 192: The messy term is

$$\begin{aligned}&\bar{u}(p')\left(-y\not{q} + z\not{p}\right)\gamma^\mu\left[(1-y)\not{q} + z\not{p}\right]u(p)\\ &\quad = \bar{u}(p')\left(z\not{p}' - (y+z)\not{q}\right)\gamma^\mu\left[(1-y)\not{q} + z\not{p}\right]u(p)\end{aligned}$$

From $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\not{q}\gamma^\mu = 2q^\mu - \gamma^\mu\not{q}$, so $\not{q}\gamma^\mu\not{q} = 2q^\mu\not{q} - \gamma^\mu q^2$. In addition, we can replace $\not{p}$ by $m$ when it acts on $u(p)$, and $\bar{u}(p')\not{p}' = \bar{u}(p')m$. Thus between $\bar{u}(p')$ and $u(p)$, $\not{q} \sim 0$, $\not{q}\gamma^\mu = \not{p}'\gamma^\mu - 2p^\mu + \gamma^\mu\not{p} \sim 2m\gamma^\mu - 2p^\mu$, and $\gamma^\mu\not{q} = 2p'^\mu - \not{p}'\gamma^\mu - \gamma^\mu\not{p} \sim 2p'^\mu - 2m\gamma^\mu$. Thus

$$\begin{aligned}&\bar{u}(p')\left(-y\not{q} + z\not{p}\right)\gamma^\mu\left[(1-y)\not{q} + z\not{p}\right]u(p)\\ &\quad = \bar{u}(p')\Big(m^2z^2\gamma^\mu - mz(1-x)\not{q}\gamma^\mu + mz(1-y)\gamma^\mu\not{q}\\ &\qquad\quad -(1-x)(1-y)(2q^\mu\not{q} - \gamma^\mu q^2)\Big)u(p)\\ &\quad = \bar{u}(p')\Big(m^2z^2\gamma^\mu - 2mz(1-x)(m\gamma^\mu - p^\mu)\\ &\qquad\quad +2mz(1-y)(p'^\mu - m\gamma^\mu) + (1-x)(1-y)\gamma^\mu q^2\Big)u(p)\\ &\quad = \bar{u}(p')\Big\{\Big(m^2(-2z - z^2) + (1-x)(1-y)q^2\Big)\gamma^\mu\\ &\qquad\quad +mz(1+z)(p'+p)^\mu + mz(x-y)q^\mu\Big]u(p).\end{aligned}$$

Adding this to the other terms in “Numerator”, the numerator is

$$\begin{aligned}&\bar{u}(p')\left[\left\{-\frac{1}{2}\ell^2 + m^2(1-2z-z^2) + (1-x)(1-y)q^2\right\}\gamma^\mu\right.\\ &\qquad\left. +mz(z-1)\left(p'^\mu + p^\mu\right) + m(z-2)(x-y)q^\mu\right]u(p)\end{aligned}$$

which is the expression cleared of smoke from the top of page 192.

Using the Gordon identity the $mz(z-1)((p'^{\mu}+p^{\mu})$ term gives $2m^2z(1-z)$ contribution to the $F_2$ piece and changes the $(1-2z-z^2)m^2$ piece in $F_1$ to $(1-4z+z^2)m^2$. Thus we have

$$\begin{aligned} F_1(q^2) &= 1+2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z) \\ &\qquad\times\frac{-\ell^2+2(1-x)(1-y)q^2+2(1-4z+z^2)m^2}{(\ell^2-\Delta+i\epsilon)^3} \\ F_2(q^2) &= +2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(1-x-y-z)\frac{4m^2z(1-z)}{(\ell^2-\Delta+i\epsilon)^3} \end{aligned}$$

with $\Delta=-xyq^2+(1-z)^2m^2>0$. These are the equivalent of 6.47.

From last time's Schwinger notes, we have (for $m-p>\frac{1}{2}$, $d<2(m-p)$)

$$\begin{aligned} I(d,p,m,\Delta) &:= \int\frac{d^d\ell}{(2\pi)^d}\frac{(\ell^2)^p}{(\ell^2-\Delta(\alpha)+i\epsilon)^m} \\ &= \frac{i\,(-1)^{m+p}\,\Gamma(\frac{1}{2}d+p)\Gamma(m-p-d/2)}{(4\pi)^{d/2}\Gamma(d/2)\Gamma(m)}\big(\Delta(\alpha)\big)^{\frac{1}{2}d+p-m} \end{aligned}$$

If we look first at the $F_2$, which comes from the $\sigma^{\mu\nu}q_\nu$ term, we need

$$\begin{aligned} \int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta+i\epsilon)^m}=I(4,0,m,\Delta) &= \frac{i(-1)^m}{(4\pi)^2}\frac{\Gamma(m-2)}{\Gamma(m)}\Delta^{2-m} \\ &= \frac{i(-1)^m}{(4\pi)^2}\frac{\Delta^{2-m}}{(m-1)(m-2)} \end{aligned}$$

because $\Gamma(z+1)=z\Gamma(z)$. This is 6.49. Note in particular for $m=3$,

$$I(4,0,3,\Delta)=\frac{-i}{2(4\pi)^2\Delta}.$$

We will see that we can get good physics from this straightforward evaluation. For the $F_1$ term, however, we also need

$$\begin{aligned} \int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^m}=I(4,1,m,\Delta) &= \frac{i(-1)^{m+1}}{(4\pi)^2}\frac{\Gamma(3)}{\Gamma(2)}\frac{\Gamma(m-3)}{\Gamma(m)}\Delta^{3-m} \\ &= \frac{i(-1)^{m+1}}{(4\pi)^2}\frac{2\Delta^{3-m}}{(m-1)(m-2)(m-3)} \end{aligned}$$

which is 6.50. Unfortunately this blows up for $m = 3$, which is where we need it.

Finally

$$\begin{aligned}
\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^3}-\frac{\ell^2}{(\ell^2-\Delta_\Lambda+i\epsilon)^3}\right)
&= \lim_{\varepsilon\to 0^-}\left(I(4+2\varepsilon,1,3,\Delta)-I(4+2\varepsilon,1,3,\Delta_\Lambda)\right)\\
&= \lim_{\varepsilon\to 0^-}\frac{i(-1)^4\Gamma(3+\varepsilon)\Gamma(-\varepsilon)}{(4\pi)^2\Gamma(2+\varepsilon)\Gamma(3)}\left[\left(\frac{\Delta}{4\pi}\right)^\varepsilon-\left(\frac{\Delta_\lambda}{4\pi}\right)^\varepsilon\right]\\
= \frac{i}{(4\pi)^2}\lim_{\varepsilon\to 0}\frac{1}{-\varepsilon}\left[1+\varepsilon\ln\frac{\Delta}{4\pi}-\left(1+\varepsilon\ln\frac{\Delta_\Lambda}{4\pi}\right)\right] &= \frac{i}{(4\pi)^2}\ln\frac{\Delta_\Lambda}{\Delta},
\end{aligned}$$

where we have used $z\Gamma(z)\underset{z\to 0}{\longrightarrow} 1$ and $z^\varepsilon \approx 1+\varepsilon\ln z$. This gives us 6.53.

After deriving 6.47 and these integrals, we see there is no difficulty in evaluating the form factor $F_2(q^2)$,

$$\begin{aligned}
F_2(q^2) &= 2ie^2\int\frac{d^4\ell}{(4\pi)^4}\int_0^1 dx\,dy\,dz\,\frac{\delta(1-x-y-z)4m^2z(1-z)}{\left(\ell^2+xyq^2-(1-z)^2m^2+i\epsilon\right)^3}\\
&= \frac{4m^2e^2}{(4\pi)^2}\int_0^1 dz\int_0^{1-z}dx\,\frac{z(1-z)}{(1-z)^2m^2-x(1-x-z)q^2}\\
&\underset{q^2\to 0}{\longrightarrow} \frac{4e^2}{(4\pi)^2}\int_0^1 dz\,\frac{z}{(1-z)}\int_0^{1-z}dx\\
&= \frac{2e^2}{(4\pi)^2}=\frac{\alpha}{2\pi}.
\end{aligned}$$

Thus we have a prediction for the "anomalous" magnetic moment:

$$a_e := \frac{g-2}{2} = F_2(0) = \frac{\alpha}{2\pi} \approx .0011614.$$

This result was found by Julian Schwinger in 1947-8, and was, I think, the first definitive result of quantum field theory. $\dfrac{\alpha}{2\pi}$ appears above his name on his tombstone.

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At the Department Colloquium on Nov. 15, 2006, we heard

New Measurement of the Electron Magnetic Moment
and the Fine Structure Constant
Gerald Gabrielse

For the first time since 1987, the magnetic moment of the electron and the fine structure constant have been measured with a greatly improved accuracy. His papers in Physical Review Letters **97** 030801-2 report

$$a_e = .001\,159\,652\,180\,85 \pm .000\,000\,000\,000\,76,$$

a measurement of $g$ of better than one part in a trillion! In order to compare these results to theory, which is now in fact used to determine $e$, they had to calculate 891 diagrams of order $\alpha^4$.

More recently, Experiment says[1]

$$\frac{g-2}{2} = \begin{array}{r} 0.001\,159\,652\,180\,73 \\ \pm 0.000\,000\,000\,000\,28 \end{array}$$

with the inverse fine structure constant

$$\alpha^{-1} = 4\pi\epsilon_0\hbar c/e^2 = \begin{array}{l} 137.035\,999\,084 \\ \pm\, 0.000\,000\,051 \end{array}$$

and theory said[2]

$$\frac{g-2}{2} = \begin{array}{rr} \alpha^4, \quad 891 \text{ diagrams} & \qquad \alpha^5, \quad 12,672 \text{ diagrams} \\ 0.001\,159\,652\,182\,79 & 0.001\,159\,652\,181\,78 \\ \pm 0.000\,000\,000\,007\,71 & \pm 0.000\,000\,000\,000\,77 \end{array}$$

certainly one of the most accurately measured quantities in physics.

The same calculations can be done for the muon[3]. The calculations in QED, of which we have given the first but in principle have provided the tools for the whole thing, need to be supplemented by the interactions with other particles with other interactions, both strong and weak. All together

$$\begin{aligned} \left(\frac{g-2}{2}\right)_{\text{exp}} &= 0.001\,165\,920\,89 \\ \left(\frac{g-2}{2}\right)_{\mu}^{\text{QED}} &= 0.001\,165\,847\,19 \end{aligned}$$

[1] D. Hanneke, S. Fogwell Hoogerheide, and G. Gabrielse, ar$\chi$iv:1009.4831; Phys. Rev. **A83** 052122 (2011).

[2] Aoyama, Hayakawa, Kinoshita, Nio, Phys. Rev. **D 77**, 053012 (2008); ar$\chi$iv:1205.5368v2

[3] Höcker and Marciano; Aoyama *et. al.* ar$\chi$iv:1205.5370v3.

$$\left(\frac{g-2}{2}\right)^{\text{EW}}_{\mu} = 0.000\,000\,001\,54$$
$$\left(\frac{g-2}{2}\right)^{\text{Had}}_{\mu} = 0.000\,000\,071\,10$$
$$\left(\frac{g-2}{2}\right)^{\text{SM}}_{\mu} = 0.001\,165\,918\,40(59)$$

so this tests all sorts of contributions of quantum field theory and the standard model.

So $F_2$ is a great success, but we also saw the integrals in $F_1$ are problematic. We will discuss that next time.