

## Lecture 10

Oct. 7, 2013

## Discrete Symmetries: P, C, and T

Two discrete transformations are needed to fill out the full Poincaré Group, that is, the transformations which preserve  $A_\mu B^\mu$ , but not in the proper orthochronous group. These can be taken to be parity  $P$ , which reverses the spacial components  $\vec{x} \rightarrow -\vec{x}$ ,  $t$  unchanged, and time reversal,  $t \rightarrow -t$ ,  $\vec{x}$  unchanged. Any element of the full group is either  $\Lambda$ ,  $P\Lambda$ ,  $T\Lambda$ , or  $PT\Lambda$ , where  $\Lambda$  is a proper orthochronous Poincaré transformation. At the same time we will consider charge conjugation  $C$ , which does nothing to space and time but interchanges particles with their antiparticles.

The behavior of various physical quantities under these transformations are defined so as to preserve as many laws of physics as possible. From the definitions, we know what happens to  $\vec{r}$  and  $t$ , and the other basic kinematic variables of a particle,  $\vec{v}$  and  $\vec{a}$  are related by time derivatives, so we know how they behave. From  $\vec{F} = m\vec{a}$ , assuming the mass  $m$  is unchanged, we see how forces in general, and the Lorentz force in particular, must behave, assuming by definition that the charge is unchanged by  $P$  and  $T$  and reversed by  $C$ . Thus we get the table of transformations shown.

$$\vec{v} = \frac{d\vec{r}}{dt}, \quad \vec{a} = \frac{d\vec{v}}{dt} \quad (1)$$

$$\vec{F} = m\vec{a} \quad (2)$$

$$\vec{F} = q\vec{E} \quad (3)$$

$$\vec{F} = q\vec{v} \times \vec{B} \quad (4)$$

$$F^{j0} = E^j \quad (5)$$

$$F^{ij} = -\epsilon_{ijk} B^k \quad (6)$$

$$\partial_\mu F^{\mu\nu} = J^\nu \quad (7)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (8)$$

|                | $P$                            | $C$            | $T$             |
|----------------|--------------------------------|----------------|-----------------|
| $\vec{r}$      | $\longrightarrow -\vec{r}$     | $\vec{r}$      | $\vec{r}$       |
| $t$            | $\longrightarrow t$            | $t$            | $-t$            |
| $\vec{v}$      | $\longrightarrow -\vec{v}$     | $\vec{v}$      | $-\vec{v}$      |
| $\vec{a}$      | $\longrightarrow -\vec{a}$     | $\vec{a}$      | $\vec{a}$       |
| $x^\mu$        | $\longrightarrow x_\mu$        | $x^\mu$        | $-x_\mu$        |
| $m$            | $\longrightarrow m$            | $m$            | $m$             |
| $\vec{F}$      | $\longrightarrow -\vec{F}$     | $\vec{F}$      | $\vec{F}$       |
| $q$            | $\longrightarrow q$            | $-q$           | $q$             |
| $\vec{E}$      | $\longrightarrow -\vec{E}$     | $-\vec{E}$     | $\vec{E}$       |
| $\vec{B}$      | $\longrightarrow \vec{B}$      | $-\vec{B}$     | $-\vec{B}$      |
| $F^{\mu\nu}$   | $\longrightarrow F_{\mu\nu}$   | $-F^{\mu\nu}$  | $-F_{\mu\nu}$   |
| $\partial_\mu$ | $\longrightarrow \partial^\mu$ | $\partial_\mu$ | $-\partial^\mu$ |
| $J^\nu$        | $\longrightarrow J_\nu$        | $-J^\nu$       | $J_\nu$         |
| $A^\mu$        | $\longrightarrow A_\nu$        | $-A^\nu$       | $A_\nu$         |

Let us remind ourselves that

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( a_p^s u^s(p) e^{-ip \cdot x} + b_p^{s\dagger} v^s(p) e^{ip \cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( b_p^s \bar{v}^s(p) e^{-ip \cdot x} + a_p^{s\dagger} \bar{u}^s(p) e^{ip \cdot x} \right)$$

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad s = 1, 2, \text{ (with } \sigma = \sigma_R, \bar{\sigma} = \sigma_L)$$

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{pmatrix} \quad s = 1, 2.$$

For 3.129-3.132, we note more generally that

$$P\bar{\psi}(t, \vec{x})\Gamma\psi(t, \vec{x})P^{-1} = \bar{\psi}(t, -\vec{x})\gamma^0\Gamma\gamma^0\psi(t, -\vec{x}). \quad (9)$$

so each spatial index gives  $-1$  and  $\gamma^5$  gives  $-1$ .

For a positronium atom at rest in a state of angular momentum  $\ell$ , we may write

$$|\psi\ell m\rangle = \int_0^\infty p^2 dp \int_0^\pi \sin\theta \int_0^{2\pi} d\phi f(p) Y_\ell^m(\theta, \phi) a_{\vec{p}}^{r\dagger} b_{-\vec{p}}^{s\dagger} |0\rangle, \quad \text{so}$$

$$P|\psi\ell m\rangle = \eta_a \eta_b \int_0^\infty p^2 dp \int_0^\pi \sin\theta \int_0^{2\pi} d\phi f(p) Y_\ell^m(\theta, \phi) a_{-\vec{p}}^{r\dagger} b_{\vec{p}}^{s\dagger} |0\rangle$$

$$= (-1) \int_0^\infty p'^2 dp' \int_0^\pi \sin\theta' \int_0^{2\pi} d\phi' f(p') Y_\ell^m(\pi - \theta', \phi' + \pi) a_{\vec{p}'}^{r\dagger} b_{-\vec{p}'}^{s\dagger} |0\rangle$$

$$= (-1)^{\ell+1} |\psi\ell m\rangle.$$

where we replaced  $\vec{p}$  with  $\vec{p}' = -\vec{p}$ , and noted that reversing the direction in the  $Y_\ell^m$  ( $\theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi$ ) just multiplies by  $(-1)^\ell$ .

If  $\hat{n}$  is a unit vector in the  $\theta, \phi$  direction,

$$\hat{n} \cdot \vec{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

Then

$$\hat{n} \cdot \vec{\sigma} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos \theta \cos(\theta/2) + \sin \theta \sin(\theta/2) \\ e^{i\phi} (\sin \theta \cos(\theta/2) - \cos \theta \sin(\theta/2)) \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{pmatrix},$$

verifying that  $\xi(\uparrow)$  given at the top of page 68 is an eigenvector of spin in the  $\hat{n}$  direction. What is being done in the middle of p. 68 is choosing the  $\eta^s$  of Eq. 3.62 for antiparticles to be  $\xi^{-s}$ , in a way which allows for an arbitrary  $\hat{n}$ .

An aside:

The somewhat arbitrary-looking definition  $\xi^{-s} = -i\sigma_2 (\xi^s)^*$  comes from the way the complex conjugate of the canonical spin  $\frac{1}{2}$  representation of rotations transforms. For any representation  $M$  of any group, the complex conjugate  $M^*$  is also a representation. For example, for SU(3) the anti-quarks are the complex conjugate representation from the quarks, and these representations are not equivalent. For SU(2), there is only one representation, up to equivalence, for each dimensionality, so the complex conjugate of a spinor must be equivalent to a spinor, and  $M^*$  must be equivalent to  $M$ . That is, there exists a single matrix  $U$  such that  $M^*(g) = UM(g)U^{-1}$  for all elements  $g \in G$  in the group. Such a representation is called a **real representation**, but it does not mean the elements of  $M$  are real. For a spinor, a rotation by  $\vec{\omega}$  is represented by  $M(\vec{\omega}) = e^{-i\omega_\ell \sigma_\ell / 2}$  so  $M^*(\vec{\omega}) = e^{i\omega_\ell \sigma_\ell^* / 2} = (-i\sigma_2)M(\vec{\omega})(-i\sigma_2)^{-1}$ . So if  $\xi^s$ , ( $s=1,2$ ), transform by the canonical spin  $\frac{1}{2}$  representation, so do  $-i\sigma_2 (\xi^s)^*$ .

A note on 3.145:

The difference between  $*$  and  $^\dagger$  is a bit confused here. I would have left off the  $-i\gamma^2 \psi^*(x)$  expression and just noted that in  $(\psi^\dagger)^T$  the transpose acts only in the four dimensional spinor space, not in the full Hilbert space in which  $\psi$  acts. The  $^\dagger$  does act in the full Hilbert space.

For the general bilinear,

$$\begin{aligned} C\bar{\psi}\Gamma\psi C^{-1} &= C\bar{\psi}_a C^{-1} \Gamma_{ab} C\psi_b C^{-1} \\ &= (-i\gamma^0 \gamma^2 \psi)_a \Gamma_{ab} (-i\bar{\psi} \gamma^0 \gamma^2)_b \\ &= (-i)^2 (-1) \bar{\psi} \gamma^0 \gamma^2 \Gamma^T \gamma^0 \gamma^2 \psi \\ &= \bar{\psi} (\gamma^0 \gamma^2)^{-1} \Gamma^T \gamma^0 \gamma^2 \psi \end{aligned}$$

Now  $(\gamma^0 \gamma^2)^{-1} \gamma^\mu \gamma^0 \gamma^2 = -\gamma^{\mu T}$ , because under transpose  $\gamma^\mu$  changes sign for  $\mu = 1, 3$  and doesn't for  $\mu = 0, 2$ , while the opposite occurs under commutation with  $\gamma^0 \gamma^2$ . Thus if  $\Gamma$  is built of  $\gamma^\mu$ 's, each one contributes a minus sign,

but also the order of the indices is reversed, which gives a minus sign if the number of  $\gamma^\mu$ 's with different  $\mu$ 's is 2 or 3 mod 4. Thus

$$\begin{aligned} C\bar{\psi}\Gamma\psi C^{-1} &= \bar{\psi}\Gamma\psi \quad \text{for} \quad \bar{\psi}\psi, \bar{\psi}\gamma_5\psi, \bar{\psi}\gamma^\mu\gamma_5\psi, \\ C\bar{\psi}\Gamma\psi C^{-1} &= -\bar{\psi}\Gamma\psi \quad \text{for} \quad \bar{\psi}\gamma^\mu\psi, \bar{\psi}\sigma^{\mu\nu}\psi. \end{aligned}$$

Please read pages 65-71.