Lecture 8 Free Dirac Particles Sept. 30, 2013

Last time we learned that we could make a Dirac field

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

with the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

arising from a Lagranian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

We learned that ψ transforms under Lorentz Transformations by

$$L_{\mu\nu} \to S_{\mu\nu} = \frac{i}{4}\left[\gamma_\mu, \gamma_\nu\right],$$

and we saw that the Dirac equation implies the Klein-Gordon equation, $(\partial^\mu \partial_\mu + m^2)\psi = 0$, though it has more restrictions on the four complex components of ψ. The matrices we are dealing with are defined by

$$\gamma^0 = \begin{pmatrix} 0_{2\times 2} & 1\!\!\mathrm{I}_{2\times 2} \\ 1\!\!\mathrm{I}_{2\times 2} & 0_{2\times 2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0_{2\times 2} & \sigma_i \\ -\sigma_i & 0_{2\times 2} \end{pmatrix}, \quad \text{or} \quad \gamma^\mu = \begin{pmatrix} 0_{2\times 2} & \sigma_R^\mu \\ \sigma_L^\mu & 0_{2\times 2} \end{pmatrix},$$

and the six generators S are then

$$S^{ij} = \frac{1}{2}\epsilon_{ijk} \begin{pmatrix} \sigma_k & 0_{2\times 2} \\ 0_{2\times 2} & \sigma_k \end{pmatrix}, \qquad S^{0j} = -\frac{i}{2} \begin{pmatrix} \sigma_j & 0_{2\times 2} \\ 0_{2\times 2} & -\sigma_j \end{pmatrix}.$$

Thus

$$\mathbf{J}_j = \frac{1}{2}\epsilon_{jk\ell}\mathbf{L}^{k\ell} \approx \frac{1}{2}\epsilon_{jk\ell}S^{k\ell} = \frac{1}{2}\begin{pmatrix} \sigma_j & 0_{2\times 2} \\ 0_{2\times 2} & \sigma_j \end{pmatrix} =: \frac{1}{2}\Sigma^j.$$

We will now continue with pages 45—51 of the text, with a few notes:

Near equations 3.49 and 3.50 it may be useful to note

$$\begin{aligned} E + p &= m(\cosh\eta + \sinh\eta) = me^\eta, \\ E - p &= m(\cosh\eta - \sinh\eta) = me^{-\eta}, \\ P_\pm &:= \left(\frac{1 \pm \sigma_3}{2}\right) \Longrightarrow P_\pm^2 = P_\pm, \quad P_\pm P_\mp = 0, \qquad \text{so} \end{aligned}$$

$$m\left(e^{\eta/2}P_- + e^{-\eta/2}P_+\right)^2 = E - \vec{p}\cdot\vec{\sigma} = p_\mu \sigma^\mu$$

and similarly, by reversing the sign of $\vec{p}$,

$$m\left(e^{\eta/2}P_+ + e^{-\eta/2}P_-\right)^2 = E + \vec{p}\cdot\vec{\sigma} = p_\mu \bar{\sigma}^\mu,$$

which justifies 3.50, for $E > |\vec{p}|$.

For 3.51, note that

$$p_\mu\sigma^\mu p_\nu\bar{\sigma}^\nu = \frac{1}{2}p_\mu p_\nu(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu) = p_0^2 - p_i^2 = p^2.$$

In saying 3.50 satifies the Dirac equation, we are replacing $\partial_\mu \to -ip_\mu$, so Eq. 3.43 becomes

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u = 0.$$

For 3.77, see "Little Note on Fierz", on the Supplementary Notes page.

On 3.78, note the upper rhs is antisymmetric under $u_{2R} \leftrightarrow u_{4R}$ which, applied to lhs, gives the last expression.

At the bottom of P49, it is worthwhile to note[1]

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\alpha\gamma^\alpha.$$

Don't forget to carefully read pages 45-51, as I will not be rewriting them!

[1]Proof: Consider the family of Lorentz transformations $\mathbf{\Lambda} = e^{-it\omega^{\mu\nu}\mathbf{L}_{\mu\nu}}$, parameterized by one real index t, If we then consider

$$f(t) := \Lambda_{\frac{1}{2}}^{-1}\gamma_\mu\Lambda^\mu{}_\alpha\Lambda_{\frac{1}{2}},$$

with $\Lambda_{\frac{1}{2}} = e^{-it\omega^{\mu\nu}S_{\mu\nu}}$ and $\Lambda^\mu{}_\alpha = \left(e^{-it\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}}\right)^\mu{}_\alpha$, we see that $f(0) = \gamma_\alpha$ and

$$\begin{aligned} \frac{df}{dt} &= e^{it\omega^{\rho\sigma}S_{\rho\sigma}}\left[i\omega^{\rho\sigma}S_{\rho\sigma}, \gamma_\mu\right]e^{-it\omega^{\rho\sigma}S_{\rho\sigma}}\left(e^{-it\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}}\right)^\mu{}_\alpha \\ &\quad + e^{it\omega^{\rho\sigma}S_{\rho\sigma}}\gamma_\mu e^{-it\omega^{\rho\sigma}S_{\rho\sigma}}\left(-i\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}\right)^\mu{}_\beta\left(e^{-it\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}}\right)^\beta{}_\alpha \end{aligned}$$

But $i\omega_{\rho\sigma}\left[S^{\rho\sigma}, \gamma_\mu\right] = -\frac{1}{2}\omega_{\rho\sigma}\left[\gamma^\rho\gamma^\sigma, \gamma_\mu\right] = -\frac{1}{2}\omega_{\rho\sigma}\gamma^\rho\left\{\gamma^\sigma, \gamma_\mu\right\} + \frac{1}{2}\omega_{\rho\sigma}\left\{\gamma^\rho, \gamma_\mu\right\}\gamma^\sigma = -\omega_{\rho\mu}\gamma^\rho$ and $\gamma_\mu\left(-i\omega^{\rho\sigma}\mathcal{L}_{\rho\sigma}\right)^\mu{}_\beta = \omega^\rho{}_\beta\gamma_\rho = \omega_{\rho\beta}\gamma^\rho$, so the two terms cancel, f is a constant γ_α. Multiplying by $\Lambda^\nu{}_\beta g^{\alpha\beta}g_{\rho\nu}$, which is the inverse of $\Lambda^\mu{}_\alpha$, gives $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\alpha\gamma^\alpha$.