Lecture 7: Dirac and Weyl Fields Sept. 26, 2013

We have seen that we expect to construct our field theory from fields which transform "simply" under Poincaré transformations, with

$$U(\Lambda)\phi_a(x)U^{-1}(\Lambda) = D_{ab}(\Lambda^{-1})\phi_b(\Lambda x), \tag{1}$$

where D is a finite dimensional representation of the Lorentz group. We also saw that such representations are in fact products of representations of two SO(3) groups generated by $\vec{L}_\pm = \frac{1}{2}(\vec{J} \pm i\vec{K})$. Thus in general there are two spins, $s_\pm$ and the field has two indices, the eigenvalues of $\vec{L}_{\pm z}$ respectively. The derivative terms of (L6 Eq. 10) can be simplified

$$\begin{aligned}
\vec{\theta}\cdot\vec{J}^\mu{}_\nu x^\nu\partial_\mu &= +\frac{1}{2}\epsilon_{ijk}\theta_i\left(\mathcal{L}_{jk}\right)^\mu{}_\nu x^\nu\partial_\mu = i\vec{\theta}\cdot(\vec{x}\times\vec{\nabla}), \\
\vec{\kappa}\cdot\vec{K}^\mu{}_\nu x^\nu\partial_\mu &= \kappa_i\left(\mathcal{L}_{0j}\right)^\mu{}_\nu x^\nu\partial_\mu = -i\vec{\kappa}\cdot\vec{x}\partial_0 - it\vec{\kappa}\cdot\vec{\nabla}.
\end{aligned}$$

Then the operators $\vec{\mathbf{J}}$ and $\vec{\mathbf{K}}$ have commutators with fields given by

$$\begin{aligned}
\left[\vec{\theta}\cdot\vec{\mathbf{J}}, \phi_{m_+,m_-}(x)\right] &= -D^A_{m_+,m'_+}(\vec{\theta}\cdot\vec{\mathbf{L}})\phi_{m'_+,m_-}(x) \\
&\quad -D^B_{m_-,m'_-}(\vec{\theta}\cdot\vec{\mathbf{L}})\phi_{m_+,m'_-}(x) \\
&\quad +i\vec{\theta}\cdot(\vec{x}\times\vec{\nabla})\phi_{m_+,m_-}(x) \\
\left[\vec{\kappa}\cdot\vec{\mathbf{K}}, \phi_{m_+,m_-}(x)\right] &= +iD^A_{m_+,m'_+}(\vec{\kappa}\cdot\vec{\mathbf{L}})\phi_{m'_+,m_-}(x) \\
&\quad -iD^B_{m_-,m'_-}(\vec{\kappa}\cdot\vec{\mathbf{L}})\phi_{m_+,m'_-}(x) \\
&\quad +-i\vec{\kappa}\cdot\vec{x}\dot{\phi}_{m_+,m_-}(x) - it\vec{\kappa}\cdot\vec{\nabla}\phi_{m_+,m_-}(x).
\end{aligned}$$

In particular, we considered a field (whose name I will now change to ψ_R, which transforms with $A = \frac{1}{2}, B = 0$, and we saw that

$$[\vec{\mathbf{J}}, \psi_{R\,m}(x)] = -\frac{1}{2}\vec{\sigma}_{mm'}\psi_{R\,m'}(x) + i\vec{x}\times\vec{\nabla}\psi_{R\,m}(x).$$

We will consider this field further, but before we do, let us also note that the Poincaré algebra $[\mathbf{L}_{\alpha\beta}, \mathbf{P}_\nu] = -ig_{\alpha\nu}\mathbf{P}_\beta + ig_{\beta\nu}\mathbf{P}_\alpha$ means

$$[\mathbf{J}_i, \mathbf{P}_j] = \frac{1}{2}\epsilon_{iab}[\mathbf{L}_{ab}, \mathbf{P}_j] = i\epsilon_{ijk}\mathbf{P}_k, \qquad [\mathbf{J}_i, \mathbf{P}_0] = 0. \tag{2}$$

$$[\mathbf{K}_i, \mathbf{P}_j] = [\mathbf{L}_{0i}, \mathbf{P}_j] = -i\delta_{ij}\mathbf{P}_0, \qquad [\mathbf{K}_i, \mathbf{P}_0] = [\mathbf{L}_{0i}, \mathbf{P}_0] = -i\mathbf{P}_i. \tag{3}$$

The $(\frac{1}{2}, 0)$ Field ψ_R

We are going to look for scalar combinations of fields, in order to construct a Lagrangian density $\mathcal{L}$. The coordinate derivative terms will work out as they should for any representation, so in what follows I am going to drop the derivative terms, with just a warning (+d.t.) that I have done so.

First suppose that ψ_R transforms with $A = \frac{1}{2}, B = 0$, so

$$[\mathbf{J}_i, \psi_R] = -\frac{1}{2}\sigma_i\psi_R, \qquad [\mathbf{K}_i, \psi_R] = +i\frac{1}{2}\sigma_i\psi_R \qquad (+\text{d.t.}).$$

Hermitean conjugate gives, as $\mathbf{J}$ and $\mathbf{K}$ are hermitean operators on the hilbert space, but the 2×2 representation of $\mathbf{K}$ is not,

$$[\mathbf{J}_i, \psi_R^\dagger] = \frac{1}{2}\psi_R^\dagger\sigma_i, \qquad [\mathbf{K}_i, \psi_R^\dagger] = +i\frac{1}{2}\psi_R^\dagger\sigma_i, \qquad (+\text{d.t.})$$

What can we make that is quadratic in ψ and its hermitian conjugate, and how do these terms transform?

$$\begin{aligned}
[\mathbf{J}_i, \psi_R^\dagger\psi_R] &= \frac{1}{2}\psi_R^\dagger\sigma_i\psi_R - \frac{1}{2}\psi_R^\dagger\sigma_i\psi_R = 0 \qquad (+\text{d.t.})\\
[\mathbf{J}_i, \psi_R^\dagger\sigma_j\psi_R] &= \frac{1}{2}\psi_R^\dagger[\sigma_i, \sigma_j]\psi_R = i\epsilon_{ijk}\psi_R^\dagger\sigma_k\psi_R \qquad (+\text{d.t.})\\
[\mathbf{K}_i, \psi_R^\dagger\psi_R] &= i\psi_R^\dagger\sigma_i\psi_R \qquad (+\text{d.t.})\\
[\mathbf{K}_i, \psi_R^\dagger\sigma_j\psi_R] &= \frac{i}{2}\psi_R^\dagger\{\sigma_i, \sigma_j\}\psi_R = i\delta_{ij}\psi_R^\dagger\psi_R \qquad (+\text{d.t.})
\end{aligned}$$

Combining with (2) we see that $\psi_R^\dagger\psi_R$, $\psi_R^\dagger\sigma_j P_j\psi_R$ and $\psi_R^\dagger P_0\psi_R$ commute with $\mathbf{J}_i$. We seek a combination which commutes with $\mathbf{K}_i$ as well.

$$\begin{aligned}
\left[K_i, \psi_R^\dagger\psi_R P_0\right] &= \left[K_i, \psi_R^\dagger\psi_R\right] P_0 + \psi_R^\dagger\psi_R\,[K_i, P_0]\\
&= i\psi_R^\dagger\sigma_i\psi_R P_0 - i\psi_R^\dagger\psi_R P_i \qquad (4)\\
\left[K_i, \sum_j \psi_R^\dagger\sigma_j\psi_R P_j\right] &= \left[K_i, \sum_j \psi_R^\dagger\sigma_j\psi_R\right] P_j + i\sum_j \psi_R^\dagger\sigma_j\psi_R\,[K_i, P_j]\\
&= i\psi_R^\dagger\psi_R P_i - i\psi_R^\dagger\sigma_i\psi_R P_0 \qquad (5)
\end{aligned}$$

so

$$\left[\mathbf{K}_i, \psi_R^\dagger\psi_R P_0 + \sum_j \psi_R^\dagger\sigma_j\psi_R P_j\right] = 0 \qquad (+\text{d.t.}).$$

The $(0, \frac{1}{2})$ Field ψ_L

On the other hand, suppose ψ_L transforms with $A = 0, B = \frac{1}{2}$, so

$$D(J_i - iK_i) = \frac{1}{2}\sigma_i,\ D(J_i + iK_i) = 0 \quad \Longrightarrow \quad D(J_i) = \frac{1}{2}\sigma_i,\ D(K_i) = +i\frac{1}{2}\sigma_i,$$

The commutations with $\mathbf{J}_i$ are therefore all the same, while the one of the fields with $\mathbf{K}_i$ are reversed, but not those of $\mathbf{K}$ with P. So now the first terms in the final expressions in (4) and (5) have their signs reversed, and the combination which is a scalar is

$$\psi_L{}^\dagger \psi_L P_0 - \sum_i \psi_L{}^\dagger \sigma_i \psi_L P_i.$$

Notice there is no invariant we can make from just ψ_R without a momentum, or from just ψ_L without a momentum, but if we mix ψ_R with ψ_L, we see $\psi_R^\dagger \psi_L$ commutes with $\mathbf{J}$ as before, and also

$$[\mathbf{K}_i, \psi_R^\dagger \psi_L] = i\frac{1}{2}\psi_R^\dagger \sigma_i \psi_L - i\frac{1}{2}\psi_R^\dagger \sigma_i \psi_L = 0,$$

so $\psi_R^\dagger \psi_L$ is an invariant. Similarly $\psi_L{}^\dagger \psi_R$ is invariant..

1 Invariant Lagrangians

The momentum transforms the same way a derivative does, so we see that the Hermitean quadratic invariants we can form from ψ_R and ψ_L are

$$\begin{aligned} & i\psi_R^\dagger \partial_0 \psi_R + i\psi_R^\dagger \vec{\sigma} \cdot \vec{\nabla} \psi_R \\ & i\psi_L^\dagger \partial_0 \psi_L - i\psi_L^\dagger \vec{\sigma} \cdot \vec{\nabla} \psi_L \\ & \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R \\ \text{and} \quad & i\psi_R^\dagger \psi_L - i\psi_L^\dagger \psi_R \end{aligned}$$

The only one which involves only ψ_R is the first, and if we vary with respect to $\psi_R^\dagger$, we get the equation of motion

$$i\partial_0 \psi_R + i\vec{\sigma} \cdot \vec{\nabla} \psi_R = 0.$$

Multiplying by $-i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})$ gives[1]

$$\begin{aligned} 0 &= (\partial_0 - \vec{\sigma}\cdot\vec{\nabla})(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = (\partial_0^2 - \sigma_i\partial_i\sigma_j\partial_j)\psi_R \\ &= (\partial_0^2 - \frac{1}{2}\{\sigma_i,\sigma_j\}\partial_i\partial_j)\psi_R = (\partial_0^2 - \delta_{ij}\partial_i\partial_j)\psi_R \\ &= (\partial_0^2 - \vec{\nabla}^{\,2})\psi_R = \partial^\mu\partial_\mu\psi_R. \end{aligned}$$

In the second line we have used the fact that $\partial_i\partial_j = \partial_j\partial_i$ to replace $\sigma_i\sigma_j$ by half the anticommutator, which we then evaluate to a Kronecker delta. We see the result is that ψ_R obeys the Klein-Gordon equation, but with zero mass. The same is true for the second lagrangian, with only ψ_L. Only by including a term with a mixture of ψ_R and ψ_L can we create a mass.

Let's define[2]

$$\sigma_R^\mu = (1, \sigma_i), \qquad \text{and} \quad \sigma_L^\mu = (1, -\sigma_i).$$

Then we can write the first two lagrangian densities as $i\psi_R^\dagger\sigma_R^\mu\partial_\mu\psi_R$ and $i\psi_L^\dagger\sigma_L^\mu\partial_\mu\psi_L$, and the equations of motion from them individually as $\sigma_R^\mu\partial_\mu\psi_R = 0$ and $\sigma_L^\mu\partial_\mu\psi_L = 0$.

If, however, we take a combination to form the lagrangian,

$$\mathcal{L} = i\psi_R^\dagger\sigma_R^\mu\partial_\mu\psi_R + i\psi_L^\dagger\sigma_L^\mu\partial_\mu\psi_L - m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R),$$

we get the equations of motion

$$\begin{aligned} i\sigma_R^\mu\partial_\mu\psi_R - m\psi_L = 0 \\ i\sigma_L^\mu\partial_\mu\psi_L - m\psi_R = 0 \end{aligned} \quad \text{or} \quad \begin{pmatrix} -m & i\sigma_R^\mu\partial_\mu \\ i\sigma_L^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0.$$

Because we are mostly interested in massive fields, we will prefer to consider ψ_L and ψ_R as parts of a four component field. Define

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \quad \text{and} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma_R^\mu \\ \sigma_L^\mu & 0\end{pmatrix}$$

[1] Properties of $\vec{\sigma}$: $\sigma_j = \sigma_j^\dagger$; $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, so $\{\sigma_i,\sigma_j\} = 2\delta_{ij}$ and $[\sigma_i,\sigma_j] = 2i\epsilon_{ijk}\sigma_k$. The usual representation, which we will assume, is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}.$$

[2] The book, and indeed everyone else who defines these, uses σ^μ for what I call σ_R^μ and $\bar{\sigma}^\mu$ for what I call σ_L^μ. But that notation is not ideal.

which means

$$\gamma^0 = \begin{pmatrix} 0 & 1\!\!\mathrm{I}_{2\times 2} \\ 1\!\!\mathrm{I}_{2\times 2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$

Then the equation above becomes

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

This is known as the Dirac equation.

A good part of learning how to calculate scattering amplitudes for fermions is becoming agile with the algebra of the γ matrices. From

$$\gamma^\mu \gamma^\nu = \begin{pmatrix} \sigma_R^\mu \sigma_L^\nu & 0 \\ 0 & \sigma_L^\mu \sigma_R^\nu \end{pmatrix}$$

we see that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times 1\!\!\mathrm{I}_{4\times 4}, \tag{6}$$

which of course means ${\gamma^0}^2 = 1$, ${\gamma^i}^2 = -1$.

Premultiplying the equation of motion $0 = (i\gamma^\mu \partial_\mu - m)\psi$ by $-(i\gamma^\nu \partial_\nu + m)$ we see that

$$\begin{aligned} 0 &= -(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = \left(\frac{1}{2}\{\gamma^\nu, \gamma^\mu\}\partial_\nu \partial_\mu + m^2\right)\psi \\ &= \left(g^{\nu\mu}\partial_\nu \partial_\mu + m^2\right)\psi = \left(\partial^\mu \partial_\mu + m^2\right)\psi, \end{aligned}$$

so the Dirac equation implies the Klein-Gordon equation with mass m, but has additional information in it.

The γ matrices will prove to be much more often used than our σ_L^μ and σ_R^μ, so we need to reexpress our Lagrangian in terms of them. Notice that $\gamma^0 \gamma^\mu = \begin{pmatrix} \sigma_L & 0 \\ 0 & \sigma_R \end{pmatrix}$ so our lagrangian can be written

$$\mathcal{L} = i\psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi - m\psi^\dagger \gamma^0 \psi.$$

That looks very strange, not even covariant, but the reason for this is that $\psi^\dagger$ does not transform as we might expect, because ψ transforms under a representation $D(\Lambda) = \Lambda_{\frac{1}{2}}$ (or $(\frac{1}{2}, 0) + (0, \frac{1}{2})$), which is **not** a unitary representation of the Lorentz group, because $\vec{L}_\pm$ involves $i\vec{K}$. Under $\psi \to \Lambda_{\frac{1}{2}}\psi$, we have $\psi^\dagger\psi \to \psi^\dagger\Lambda_{\frac12}^\dagger\Lambda_{\frac12}\psi$, and if $\Lambda_{\frac12}$ were unitary we would have $\Lambda_{\frac12}^\dagger = \Lambda_{\frac12}^{-1}$, and $\psi^\dagger\psi$ would be invariant, as it appears. But this is not the case.

What is $\Lambda_{\frac12}$? It turns out there is a simple expression for its generators in terms of

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right].$$

From the anticommutation relations of the gammas (6) simple algebraic manipulations show that $D(L^{\mu\nu}) = S^{\mu\nu}$ obeys the Lorentz algebra commutation relations, and thus is a representation. In fact, from

$$S^{ij} = \frac12\epsilon_{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}, \quad S^{0j} = -\frac{i}{2}\begin{pmatrix}\sigma_j & 0\\ 0 & -\sigma_j\end{pmatrix},$$

we see that this is exactly how ψ transforms, or rather that

$$\Lambda_{\frac12}\left(e^{-\frac{i}{2}\omega^{\mu\nu}\mathbf{L}_{\mu\nu}}\right) = e^{-\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}}.$$

Now notice that $\gamma_0^{-1}\gamma^\mu\gamma_0 = \gamma_\mu = (\gamma^\mu)^\dagger$, which means that $\gamma_0^{-1}S_{\mu\nu}^\dagger\gamma_0 = S_{\mu\nu}$ and $\gamma_0^{-1}\Lambda_{\frac12}^\dagger\gamma_0 = \Lambda_{\frac12}^{-1}$. Thus if we define $\bar\psi := \psi^\dagger\gamma_0$, under a Lorentz transformation

$$\bar\psi \to \psi^\dagger\Lambda_{\frac12}^\dagger\gamma_0 = \psi^\dagger\gamma_0\gamma_0^{-1}\Lambda_{\frac12}^\dagger\gamma_0 = \bar\psi\Lambda_{\frac12}^{-1},$$

so $\bar\psi\psi$ is invariant, and so is $\bar\psi\gamma^\mu\partial_\mu\psi$. Thus we can rewrite the free Dirac lagrangian density as

$$\mathcal{L}_{\rm Dirac} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi.$$