Lecture 6: The Poincaré Group Sept. 23, 2013

Last time we saw that for a scalar field $\phi(x)$, for every Poincaré transformation $\Lambda : x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + c^\mu$, there is a unitary operator $U(\Lambda)$ which transforms the field by $U(\Lambda)\phi(x)U^{-1}(\Lambda) = \phi(\Lambda x)$. As for the individual particle states, we expect there may be sets of fields which, rather than transforming as scalars, transform within themselves, with

$$U(\Lambda)\phi_a(x)U^{-1}(\Lambda) = M_{ba}(\Lambda)\,\phi_b(\Lambda x).$$

Two successive Poincaré transformations must transform as their composite,

$$\begin{aligned} U(\Lambda_2\Lambda_1)\,\phi_a(x)\,U^{-1}(\Lambda_2\Lambda_1) &= M_{ca}(\Lambda_2\Lambda_1)\,\phi_c(\Lambda_2\Lambda_1 x) \\ &= U(\Lambda_2)\,U(\Lambda_1)\,\phi_a(x)\,U^{-1}(\Lambda_1)\,U^{-1}(\Lambda_2) \\ &= U(\Lambda_2)\,M_{ba}(\Lambda_1)\,\phi_b(\Lambda_1 x)\,U^{-1}(\Lambda_2) \\ &= M_{cb}(\Lambda_2)M_{ba}(\Lambda_1)\,\phi_c(\Lambda_2\Lambda_1 x) \end{aligned}$$

which requires that $\Lambda \mapsto M(\Lambda)$ is a representation,

$$M_{ca}(\Lambda_2\Lambda_1) = M_{cb}(\Lambda_2)\,M_{ba}(\Lambda_1),$$

in the same way as we found for the action on states.

So to discuss fields like the electromagnetic field, which we expect will not transform like a scalar, we need to understand the possible representations of the Poincaré group, and in particular the Lorentz subgroup. So it is time to discuss this group in detail.

For particle theorists a primary requirement of a quantum field theory is that it be invariant under the Poincaré group, which consists of the proper orthochronous Lorentz transformations and the translations:

$$\Lambda : x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + c^\mu, \tag{1}$$

with c^μ an arbitrary constant vector. The Lorentz condition on the real matrix Λ is that it preserve the Minkowski product: if $V'^\mu = \Lambda^\mu{}_\nu V^\nu$ and $W'^\mu = \Lambda^\mu{}_\nu W^\nu$, then

$$\begin{aligned} V^\mu W_\mu &= V'^\mu W'_\mu = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma V^\rho W^\sigma \\ &= g_{\rho\sigma} V^\rho W^\sigma \end{aligned}$$

for any two vectors V and W, from which we can conclude

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}. \tag{2}$$

In matrix language this is $\Lambda^T g\Lambda = g$, a pseudo-orthogonality condition, where g replaces the identity in the usual orthogonality condition $\mathcal{O}^T\mathcal{O} = 1\!\!1$, with $\mathcal{O}^T$ the transpose of the matrix $\mathcal{O}$.

Examining the (00) component of this equation, we have

$$(\Lambda^0{}_0)^2 - \sum_{j=1}^{3}(\Lambda^j{}_0)^2 = 1$$

we see that, because $\Lambda^\mu{}_\nu$ is real, $(\Lambda^0{}_0)^2 \geq 1$. This divides the Lorentz transformations into those with $\Lambda^0{}_0 \geq 1$, which are called **orthochronous** because they preserve the direction of time, and those with negative $\Lambda^0{}_0$, which do not. We may also take the determinant of (2) to conclude $(\det\Lambda)^2 = 1$, so this divides the Lorentz transformations into those with determinant $+1$ and those with determinant -1. Only the **proper orthochronous** Lorentz transformations, those with positive $\Lambda^0{}_0$ and positive determinant, can arise from a continuous acceleration or rotation, and it is only these which are essential for any high energy theory. The others involve parity or time-reversal, and are still interesting, but we will delay discussion of them.

Any proper orthochronous Lorentz transformation can be written as the repeated application of an infinitesimal one. Thus we can write $\Lambda = e^{a_\ell \tilde{L}_\ell}$, where a_ℓ are some continuous real parameters describing the group element, and each $\tilde{L}_\ell{}^{\cdot}{}_{\cdot}$ is a real 4×4 real matrix. As $g_{\nu\tau}$ is a constant, differentiating (2) with respect to a_ℓ at $a_\ell = 0$ gives

$$g_{\mu\nu}\tilde{L}_\ell{}^\mu{}_\rho\delta^\nu_\sigma + g_{\mu\nu}\delta^\mu_\rho\tilde{L}_\ell{}^\nu{}_\sigma = 0, \quad \text{or } \tilde{L}_{\ell\,\sigma\rho} + \tilde{L}_{\ell\,\rho\sigma} = 0,$$

that is, $\tilde{L}_{\ell\,\rho\sigma}$ is a real antisymmetric matrix. There are six linearly independent 4×4 antisymmetric real matrices, corresponding to the three rotations and three directions for Lorentz boosts, and thus there are 6 components to a_ℓ.

For a continuous (Lie) group which is connected, as the proper orthochronous Lorentz transformation group is, most[1] of the group properties are

[1]We will see later that some global properties are not determined. An example you already know comes from the rotation group SO(3) and the group of unitary 2×2 matrices of determinant 1, SU(2). The generators of these have the same familiar algebra, $[L_i, L_j] = i\epsilon_{ijk}L_k$, but a rotation through 2π gives the identity for SO(3) but it takes a rotation by 4π to reach the identity in SU(2). The latter is called the *covering group* of SO(3).

determined by the commutation relations of the generators, that is, by the derivatives of the group elements with respect to the group parameters, evaluated at the identity element. The generators form a vector space which is a Lie algebra. In our case, that means we will learn what we need from the commutators of the $\tilde{L}_\ell$ matrices, together with their commutators with the translations.

The index ℓ describing the six independant antisymmetric 4×4 real matrices is most conveniently described by a pair of 4 dimensional indices, with the understanding that $\tilde{L}_{\alpha\beta} = -\tilde{L}_{\beta\alpha}$. Please note that $\tilde{L}_{\alpha\beta}$, for each pair α,β, is a matrix, not a matrix element. We may define a basis for the vector space of generators

$$\left(\tilde{L}_{\alpha\beta}\right)^{\mu\nu} = \delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha.$$

Matrix multiplication requires lowering one matrix element index,

$$\left(\tilde{L}_{\alpha\beta}\right)^{\mu}{}_{\nu} = \delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu}$$

and the commutator is therefore

$$\begin{aligned}
\left[\tilde{L}_{\alpha\beta},\tilde{L}_{\gamma\zeta}\right]^{\mu}{}_{\nu} &= \left(\tilde{L}_{\alpha\beta}\right)^{\mu}{}_{\rho}\left(\tilde{L}_{\gamma\zeta}\right)^{\rho}{}_{\nu} - \left(\tilde{L}_{\gamma\zeta}\right)^{\mu}{}_{\rho}\left(\tilde{L}_{\alpha\beta}\right)^{\rho}{}_{\nu} \\
&= \left(\delta^\mu_\alpha g_{\beta\rho} - \delta^\mu_\beta g_{\alpha\rho}\right)\left(\delta^\rho_\gamma g_{\zeta\nu} - \delta^\rho_\zeta g_{\gamma\nu}\right) \\
&\quad - \left(\delta^\mu_\gamma g_{\zeta\rho} - \delta^\mu_\zeta g_{\gamma\rho}\right)\left(\delta^\rho_\alpha g_{\beta\nu} - \delta^\rho_\beta g_{\alpha\nu}\right) \\
&= \delta^\mu_\alpha g_{\beta\rho}\delta^\rho_\gamma g_{\zeta\nu} - \delta^\mu_\alpha g_{\beta\rho}\delta^\rho_\zeta g_{\gamma\nu} - \delta^\mu_\beta g_{\alpha\rho}\delta^\rho_\gamma g_{\zeta\nu} + \delta^\mu_\beta g_{\alpha\rho}\delta^\rho_\zeta g_{\gamma\nu} \\
&\quad -\delta^\mu_\gamma g_{\zeta\rho}\delta^\rho_\alpha g_{\beta\nu} + \delta^\mu_\gamma g_{\zeta\rho}\delta^\rho_\beta g_{\alpha\nu} + \delta^\mu_\zeta g_{\gamma\rho}\delta^\rho_\alpha g_{\beta\nu} - \delta^\mu_\zeta g_{\gamma\rho}\delta^\rho_\beta g_{\alpha\nu} \\
&= \delta^\mu_\alpha g_{\beta\gamma} g_{\zeta\nu} - \delta^\mu_\alpha g_{\beta\zeta} g_{\gamma\nu} - \delta^\mu_\beta g_{\alpha\gamma} g_{\zeta\nu} + \delta^\mu_\beta g_{\alpha\zeta} g_{\gamma\nu} \\
&\quad -\delta^\mu_\gamma g_{\zeta\alpha} g_{\beta\nu} + \delta^\mu_\gamma g_{\zeta\beta} g_{\alpha\nu} + \delta^\mu_\zeta g_{\gamma\alpha} g_{\beta\nu} - \delta^\mu_\zeta g_{\gamma\beta} g_{\alpha\nu} \\
&= g_{\alpha\gamma}\left(\delta^\mu_\zeta g_{\beta\nu} - \delta^\mu_\beta g_{\zeta\nu}\right) - g_{\beta\gamma}\left(\delta^\mu_\zeta g_{\alpha\nu} - \delta^\mu_\alpha g_{\zeta\nu}\right) \\
&\quad -g_{\alpha\zeta}\left(\delta^\mu_\gamma g_{\beta\nu} - \delta^\mu_\beta g_{\gamma\nu}\right) + g_{\zeta\beta}\left(\delta^\mu_\gamma g_{\alpha\nu} - \delta^\mu_\alpha g_{\gamma\nu}\right) \\
&= -g_{\alpha\gamma}\left(\tilde{L}_{\beta\zeta}\right)^{\mu}{}_{\nu} + g_{\beta\gamma}\left(\tilde{L}_{\alpha\zeta}\right)^{\mu}{}_{\nu} + g_{\alpha\zeta}\left(\tilde{L}_{\beta\gamma}\right)^{\mu}{}_{\nu} - g_{\beta\zeta}\left(\tilde{L}_{\alpha\gamma}\right)^{\mu}{}_{\nu}
\end{aligned} \tag{3}$$

As must be the case, the commutator of two generators is a linear combination of generators, for any Lie algebra is closed under commutation. We have written things as mathematicians do, but physicists generally prefer to write their group elements as

$$\Lambda = e^{-i\sum_\ell a_\ell L_\ell},$$

with the extra $-i$ in the exponent, because they expect the group transformation to act as a unitary operator and would like the generators to act as hermitean operators. We may therefore write the physicists' generators as $\mathcal{L}_{\alpha\beta} = i\tilde{L}_{\alpha\beta}$, with

$$(\mathcal{L}_{\alpha\beta})^\mu{}_\nu = i\delta^\mu_\alpha g_{\beta\nu} - i\delta^\mu_\beta g_{\alpha\nu},$$

$$[\mathcal{L}_{\alpha\beta}, \mathcal{L}_{\gamma\zeta}]^\mu{}_\nu = -ig_{\alpha\gamma}\,(\mathcal{L}_{\beta\zeta})^\mu{}_\nu + ig_{\beta\gamma}\,(\mathcal{L}_{\alpha\zeta})^\mu{}_\nu + ig_{\alpha\zeta}\,(\mathcal{L}_{\beta\gamma})^\mu{}_\nu - ig_{\zeta\beta}\,(\mathcal{L}_{\alpha\gamma})^\mu{}_\nu$$

or as matrices,

$$[\mathcal{L}_{\alpha\beta}, \mathcal{L}_{\gamma\zeta}] = -ig_{\alpha\gamma}\,\mathcal{L}_{\beta\zeta} + ig_{\beta\gamma}\,\mathcal{L}_{\alpha\zeta} + ig_{\alpha\zeta}\,\mathcal{L}_{\beta\gamma} - ig_{\zeta\beta}\,\mathcal{L}_{\alpha\gamma} \tag{4}$$

We will not be needing the $\tilde{L}_{\alpha\beta}$ any more, so from now on the $\tilde{}$ will be reserved for other meanings.

Because sometimes we think in terms of space and time, and not always in four dimensions, it is also useful to divide the six generators of the general Lorentz transformations into three spatial ones[2] and the three space-time ones:

$$J_\ell = \frac{1}{2}\epsilon_{\ell jk}\mathcal{L}_{jk}, \qquad K_j = \mathcal{L}_{0j}.$$

We can find their commutators from (4), keeping in mind that $g_{ij} = -\delta_{ij}$, and noting that $\mathcal{L}_{jk} = \epsilon_{jk\ell}J_\ell$:

$$\begin{aligned}
[J_j, J_k] &= \frac{1}{4}\epsilon_{j\ell m}\epsilon_{kpq}\,[\mathcal{L}_{\ell m}, \mathcal{L}_{pq}] \\
&= \frac{i}{4}\epsilon_{j\ell m}\epsilon_{kpq}\,(\delta_{\ell p}\mathcal{L}_{mq} - \delta_{mp}\mathcal{L}_{\ell q} - \delta_{\ell q}\mathcal{L}_{mp} + \delta_{mq}\mathcal{L}_{\ell p}) \\
&= \frac{i}{4}\left(\epsilon_{jpm}\epsilon_{kpq}\mathcal{L}_{mq} - \epsilon_{j\ell p}\epsilon_{kpq}\mathcal{L}_{\ell q} - \epsilon_{j\ell m}\epsilon_{kp\ell}\mathcal{L}_{mp} + \epsilon_{j\ell m}\epsilon_{kpm}\mathcal{L}_{\ell p}\right) \\
&= -\frac{i}{4}\Big([-\delta_{jk}\delta_{mq} + \delta_{mk}\delta_{jq}]\mathcal{L}_{mq} + [\delta_{jq}\delta_{\ell k} - \delta_{\ell q}\delta_{jk}]\mathcal{L}_{\ell q} \\
&\qquad + [\delta_{mk}\delta_{jp} - \delta_{jk}\delta_{mp}]\mathcal{L}_{mp} - [\delta_{jk}\delta_{\ell p} - \delta_{\ell k}\delta_{jp}]\mathcal{L}_{\ell p}\Big)
\end{aligned}$$

[2]If you need a refresher (or remediation) on the Levi-Civita tensor ϵ_{ijk}, please see the the supplementary note on "ϵ_{ijk} and Cross Products in 3-D Euclidean space".

$$\begin{aligned}
&= -\frac{i}{4}\left(\mathcal{L}_{kj} + \mathcal{L}_{kj} + \mathcal{L}_{kj} + \mathcal{L}_{kj}\right) \\
&= i\,\mathcal{L}_{jk} = i\epsilon_{jk\ell}J_\ell. \qquad (5) \\
[J_j, K_k] &= \frac{1}{2}\epsilon_{jpq}[\mathcal{L}_{pq}, \mathcal{L}_{0k}] = \frac{i}{2}\epsilon_{jpq}\left(\delta_{pk}\mathcal{L}_{0q} - \delta_{qk}\mathcal{L}_{0p}\right) = +i\epsilon_{jkq}\mathcal{L}_{0q} \\
[J_j, K_k] &= i\epsilon_{jkq}K_q. \qquad (6) \\
[K_j, K_k] &= [\mathcal{L}_{0j}, \mathcal{L}_{0k}] = -i\mathcal{L}_{jk} \\
&= -i\epsilon_{jk\ell}J_\ell. \qquad (7)
\end{aligned}$$

Eqs. (5) and (6) are the usual commutation relations for the rotation operator with any vector, $[J_j, V_k] = i\epsilon_{jk\ell}V_\ell$. So $\vec{J}$ and $\vec{K}$ rotate as vectors ought to, under the action of the angular momentum generators $\vec{J}$. Eq. (7) is something else however, a somewhat surprising statement that Lorentz boosts do not commute but rather their commutator is a generator of a rotation. (In 504, E&M II, we see that this gives rise to the Thomas precession and $g = 2$ for the electron.)

The algebra of these six generators is simplified if we consider the complex linear combinations $L_{j\pm} := \frac{1}{2}(J_j \pm iK_j)$, which satisfy the commutators

$$\begin{aligned}
[L_{j+}, L_{k+}] &= \frac{i}{4}\epsilon_{jk\ell}\left(J_\ell + iK_\ell + iK_\ell + J_\ell\right) = i\epsilon_{jk\ell}L_{\ell+}, \\
[L_{j+}, L_{k-}] &= \frac{i}{4}\epsilon_{jk\ell}\left(J_\ell - iK_\ell + iK_\ell - J_\ell\right) = 0, \\
[L_{j-}, L_{k-}] &= \frac{i}{4}\epsilon_{jk\ell}\left(J_\ell - iK_\ell - iK_\ell + J_\ell\right) = i\epsilon_{jk\ell}L_{\ell-}. \qquad (8)
\end{aligned}$$

Thus we have two sets of mutually commuting generators, so we can find the possible representations of fields by asking how they transform under each of the two independant algebras, each of which has the commutation relations of ordinary rotations, SO(3) or SU(2). We know the finite dimensional representations from our quantum mechanics course — they are labelled by a total spin which is a half integer.

We have discussed the properties of the Lorentz transformations as if they were simply matrices acting on coordinates, but of course we also have operators which act on the states of our system, provided for us by the Noether theorem. These operators need to have the same group properties as matrices do, but to distinguish the more general operators, we will write them in boldface, $\mathbf{\Lambda}, \mathbf{L}_{\alpha\beta}, \mathbf{J}_\ell, \mathbf{K}_\ell$, and $\mathbf{L}_{\ell\pm}$ being the operator versions of $\Lambda, \mathcal{L}_{\alpha\beta}, J_\ell, K_\ell$, and $L_{\ell\pm}$ respectively. We need to make this distinction at this point to deal with translations, which do not act as matrices on the coordinate space.

1 Including the translations

The full Poincaré group contains, in addition to the Lorentz transformations, translations $\mathbf{T}$ in four dimensions, generated by the 4-momentum operator $\mathbf{P}^\mu$,

$$\mathbf{T} = e^{-ic^\mu \mathbf{P}_\mu} : x^\mu \to x'^\mu = x^\mu + c^\mu.$$

Clearly translations by amounts c_1^μ and c_2^μ commute with each other, so

$$[\mathbf{P}^\mu, \mathbf{P}^\nu] = 0.$$

But the translations do not commute with the Lorentz transformations: $(\mathbf{\Lambda}\,\mathbf{T}x)^\mu = \left(e^{-ia_\ell \mathcal{L}_\ell}\right)^\mu{}_\nu (x^\nu + c^\nu)$, while $(\mathbf{T}\,\mathbf{\Lambda}x)^\mu = \left(e^{-ia_\ell \mathcal{L}_\ell}\right)^\mu{}_\nu x^\nu + c^\mu$, so

$$([\mathbf{\Lambda}, \mathbf{T}]x)^\mu = \left(e^{-ia_\ell \mathcal{L}_\ell} - 1\!\!1\right)^\mu{}_\nu c^\nu.$$

As $e^{-ic^\mu \mathbf{P}_\mu}$ is the operator which implements $T : x^\mu \to x'^\mu = x^\mu + c^\mu$, we have

$$\left[e^{-ia_\ell \mathbf{L}_\ell}, e^{-ic^\mu \mathbf{P}_\mu}\right] = e^{-i\left(e^{-ia_\ell \mathcal{L}_\ell}\right)^\mu{}_\nu c^\nu \mathbf{P}_\mu} - e^{-ic^\mu \mathbf{P}_\mu}.$$

Expanding to first order in a_ℓ and c_ν gives

$$[\mathbf{L}_{\alpha\beta}, \mathbf{P}_\nu] = (\mathcal{L}_{\alpha\beta})^\mu{}_\nu \mathbf{P}_\mu = i(\delta^\mu_\alpha g_{\beta\nu} - \delta^\mu_\beta g_{\alpha\nu})\mathbf{P}_\mu = -ig_{\alpha\nu}\mathbf{P}_\beta + ig_{\beta\nu}\mathbf{P}_\alpha.$$

Thus we have found the commutation relations which define the Lie algebra of the Poincaré group.

2 Casimir Operators

An operator $\mathbf{C}$ constructed from the Lie algebra generators which commutes with all the generators is called a *Casimir* operator. One easy example for the Poincaré group is $\mathbf{C}_1 = \mathbf{P}^2 := \mathbf{P}^\mu \mathbf{P}_\mu$, for it obviously commutes with all $\mathbf{P}^\nu$, but also

$$\begin{aligned}
[\mathbf{L}_{\alpha\beta}, \mathbf{C}_1] &= [\mathbf{L}_{\alpha\beta}, \mathbf{P}^\mu]\mathbf{P}_\mu + \mathbf{P}^\mu[\mathbf{L}_{\alpha\beta}, \mathbf{P}_\mu] \\
&= i\delta^\mu_\alpha \mathbf{P}_\beta \mathbf{P}_\mu - i\delta^\mu_\beta \mathbf{P}_\alpha \mathbf{P}_\mu + ig_{\alpha\mu}\mathbf{P}^\mu \mathbf{P}_\beta - ig_{\beta\mu}\mathbf{P}^\mu \mathbf{P}_\alpha \\
&= i[\mathbf{P}_\beta, \mathbf{P}_\alpha] + i[\mathbf{P}_\alpha, \mathbf{P}_\beta] = 0
\end{aligned}$$

as the $\mathbf{P}$'s commute. Actually, as you will find useful to prove for Homework #3, question 1 , the Lorentz product of any two vectors $W^\mu V_\mu$ commutes with $\mathbf{L}_{\alpha\beta}$, even if W and V do not commute with each other.

A less obvious Casimir operator is the square of the four-vector

$$\mathbf{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathbf{P}_\nu \mathbf{L}_{\rho\sigma},$$

which is known as the *Pauli-Lubanski* vector. Here $\epsilon^{\mu\nu\rho\sigma}$ is the four-dimensional Minkowski space tensor, defined by being totally antisymmetric under interchange of any two indices, with $\epsilon^{0123} = 1$. In Homework #3, question 1 we show that $\mathbf{W}^2$ also commutes with all the generators of the Poincaré group.

Because any Casimir operator commutes with all the generators of the group, any irreducible representation has the Casimir operator acting as a c-number on it. As we have seen, we expect single particle states to lie in irreducible representations of the Poincaré group, so to have specific numerical values for $\mathbf{P}^2$ and $\mathbf{W}^2$.

Of course we recognize $P^2 = E^2 - \vec{P}^{\,2} = m^2$ as the square of the mass of a system. What is W^2? This is most easily understood classically by going to rest frame of the system, where $\vec{P} = 0$, $P^0 = m$. As $\mathbf{W}^2$ is Lorentz invariant (it commutes with all $\mathbf{L}_{\mu\nu}$) this is sufficient. Then $\mathbf{W}^\mu = \frac{1}{2}m\,\epsilon^{\mu 0\rho\sigma}\,\mathbf{L}_{\rho\sigma} = \frac{1}{2}m\,\epsilon^{\mu 0jk}\,\mathbf{L}_{jk}$, which vanishes for $\mu = 0$ and $\mathbf{W}^\ell = -\frac{1}{2}m\,\epsilon_{\ell jk}\,\mathbf{L}_{jk} = -m\mathbf{J}_\ell$, so $W^2 = -m^2 J^2 = -m^2 s(s+1)$, where s is the quantum number for the total angular momentum of the system in its rest frame. For a single particle that is called the spin, and from quantum mechanics we know that s must take on only half-integer values.

The above argument assumed our state had a positive P^2. We might wish to exclude from consideration states with $P^2 < 0$, which are tachyons moving faster than the speed of light, for which there are at least some tricky problems in being consistent with causality and relativity, and for which there is no experimental evidence, despite recent excitement. But we certainly cannot exclude massless particle states with $P^2 = 0$. Note that in general $P_\mu W^\mu = 0$ from the definition of W and the commutation of the momentum. For massless states which might be the limit of those with mass without having $s \to \infty$, we will have[3] $W^2 = 0$. Then

$$\mathbf{W}_\mu \mathbf{W}^\mu \left|p\right\rangle = \mathbf{W}_\mu \mathbf{P}^\mu \left|p\right\rangle = \mathbf{P}_\mu \mathbf{P}^\mu \left|p\right\rangle = 0.$$

[3]States with $m = 0$, $W^2 \neq 0$ do not seem to occur, though I am not sure what would follow for them.

The only null vectors whose products with a given non-zero null vector vanish are those proportional to that vector. To check that, go to the frame where the given vector is $(E, 0, 0, \pm E)$. Then the other null vector V has $V_0 = \pm V_3$ and has no room for V_1 or V_2. So $\mathbf{W}_\mu |p\rangle = h\mathbf{P}_\mu |p\rangle$ for some number h, known as the *helicity*. As $\mathbf{W}$ and $\mathbf{P}$ both transform as vectors under proper Lorentz transformations, we expect h to be invariant, and by examining its value in the reference frame with $P^\mu = (E, 0, 0, E)$ we see that the helicity is the angular momentum in the direction of motion,

$$h = -\frac{\vec{J} \cdot \vec{P}}{|\vec{P}|}.$$

But $\vec{J}$ is a pseudo-vector, while $\vec{P}$ is a vector, that is, under parity $\vec{x} \to -\vec{x}$, $t \to +t$, $\vec{P}$ changes sign, while $\vec{J}$ does not. So the helicity changes sign under parity, and if we want a parity-invariant theory, our non-zero helicity states must occur in pairs, while if we don't care about parity that is not the case. We shall see that this is an important issue in neutrino physics.

As we saw earlier, a field that is not a scalar will be part of a collection $\phi_a(x)$ satisfying $U(\Lambda)\phi_a(x)U^{-1}(\Lambda) = M_{ba}(\Lambda)\phi_b(\Lambda x)$ for some representation M of the Lorentz group. The books all write this differently,

$$U(\Lambda)\phi_a(x)U^{-1}(\Lambda) = D_{ab}(\Lambda^{-1})\phi_b(\Lambda x), \tag{9}$$

where D is a representation, but that is equivalent[4].

Now, as we have seen, the Lie algebra of the Lorentz transformations can be broken up into two commuting SU(2) algebras. We know that the finite

[4]Here are some simple facts about representations:
If $M : G \to N \times N$ complex matrices is a representation, (so $M(g_1)M(g_2) = M(g_1 \circ g_2)$), so are

$$\begin{aligned} &M_C : g \mapsto (M(g))^* \\ &M_I : g \mapsto (M(g^{-1}))^{\mathrm{T}} \\ &M_H : g \mapsto (M(g^{-1}))^\dagger \end{aligned}$$

For example,

$$\begin{aligned} M_I(g_1)M_I(g_2) &= (M(g_1^{-1}))^{\mathrm{T}}(M(g_2^{-1}))^{\mathrm{T}} = (M(g_2^{-1})M(g_1^{-1}))^{\mathrm{T}} = (M(g_2^{-1} \circ g_1^{-1}))^{\mathrm{T}} \\ &= (M((g_1 \circ g_2)^{-1}))^{\mathrm{T}} = M_I(g_1 \circ g_2). \end{aligned}$$

In particular, by replacing the representation $M(g)$ by $D(g) = M_I(g) = M(g^{-1})^{\mathrm{T}}$, we get the revised expression (9).

dimensional representations of SU(2) are labelled by a half integer spin s, and the two subscript indices are the m_z indices which run from $-s$ to s by unit steps. Each index for the state under Lorentz transformations then turns into a pair of indices, one for L_+ and one for L_-. We will call the spin A for L_+ and B for L_-, and replace ϕ_b by $\phi_{m_1 m_2}$. Rewriting

$$\begin{aligned}\mathbf{\Lambda} &= e^{-i\vec{\theta}\cdot\vec{\mathbf{J}} - i\vec{\kappa}\cdot\vec{\mathbf{K}}} = e^{-i\vec{\theta}\cdot(\vec{\mathbf{L}}_+ + \vec{\mathbf{L}}_-) - \vec{\kappa}\cdot(\vec{\mathbf{L}}_+ - \vec{\mathbf{L}}_-)} \\ &= e^{-i\vec{\theta}_+\cdot\vec{\mathbf{L}}_+ - i\vec{\theta}_-\cdot\vec{\mathbf{L}}_-}\end{aligned}$$

with

$$\theta_+ = \theta - i\kappa, \qquad \theta_- = \theta + i\kappa.$$

Then

$$\begin{aligned}&D_{(m_1 m_2)(m_1' m_2')}\left(e^{+i\vec{\theta}_+\cdot\vec{\mathbf{L}}_+ + i\vec{\theta}_-\cdot\vec{\mathbf{L}}_-}\right) \\ &\quad = D^A_{m_1 m_1'}\left(e^{+i\vec{\theta}_+\cdot\vec{\mathbf{L}}_+}\right) D^B_{m_2 m_2'}\left(e^{+i\vec{\theta}_-\cdot\vec{\mathbf{L}}_-}\right).\end{aligned}$$

Evaluating $U(\Lambda)\phi_a(x)U^{-1}(\Lambda) - \phi_a(x)$ from (9) to first order, we have[5]

$$\begin{aligned}-i\left[\vec{\theta}\cdot\vec{\mathbf{J}} + \vec{\kappa}\cdot\vec{\mathbf{K}}, \phi_{m_1 m_2}(x)\right] &= D^A_{m_1 m_1'}(i\vec{\theta}_+\cdot\vec{\mathbf{L}}_+)\phi_{m_1' m_2} \\ &+ D^B_{m_2 m_2'}(i\vec{\theta}_-\cdot\vec{\mathbf{L}}_-)\phi_{m_1 m_2'} + \left(-i\vec{\theta}\cdot\vec{J}^{\mu}{}_{\nu} - i\vec{\kappa}\cdot\vec{K}^{\mu}{}_{\nu}\right)x^{\nu}\partial_\mu\phi_{m_1 m_2}.\end{aligned} \tag{10}$$

Let us first consider the $(\frac{1}{2}, 0)$ representation, that is, $A = \frac{1}{2}$, $B = 0$. Then there are no m_2 indices (or rather, there is only one value for it) and $D^0 = 1$, while

$$D^{\frac{1}{2}}(\mathbf{L_i}) = \frac{1}{2}\sigma_i,$$

the familiar Pauli spin matrix. Thus

$$\begin{aligned}[\mathbf{J}_\ell, \phi_m(x)] &= -\frac{1}{2}\sigma_{\ell\, mm'}\phi_{m'} + J_\ell{}^{\mu}{}_{\nu}x^{\nu}\partial_\mu\phi_m \\ &= -\frac{1}{2}\sigma_{\ell\, mm'}\phi_{m'} - i\epsilon_{\ell jk}x^k\partial_j\phi_m\end{aligned}$$

[5] We are using the same notation D for the representation of the group elements, $D: g \to D(g)$ for $g \in G$, and for the representation of the generators of infinitesimal transformations, $L \in \mathcal{G}$, where the **Lie algebra** $\mathcal{G}$ is the set of linear combinations of the basis of generators. Hopefully you have already mastered this possibly confusing notation in treating the rotation group in quantum mechanics. As the group elements $g = e^{i\theta_\ell L_\ell}$ can all be written as exponentials of the Lie algebra elements L_ℓ, it is natural to define $D(g) = e^{i\theta_\ell D(L_\ell)}$.

or

$$\left[\vec{\mathbf{J}}, \phi_m(x)\right] = -\frac{1}{2}\vec{\sigma}_{mm'}\phi_{m'} + i\vec{x} \times \vec{\nabla}\phi_m.$$

Notice that this is perhaps opposite of what you would expect from non-relativistic quantum mechanics, as we might have expected

$$\vec{L} = \vec{r} \times \vec{p} + \frac{1}{2}\vec{\sigma}, \qquad \text{(with } \vec{p} = -i\hbar\vec{\nabla}\text{)}$$

for a spin 1/2 particle. But our expressions for H and P had the same reversed sign,

$$\begin{aligned} [H, \phi] &= -i\frac{\partial}{\partial t}\phi, \\ [P_j, \phi] &= +i\frac{\partial}{\partial x^j}\phi, \end{aligned}$$

which are similarly opposite to $H = i\hbar\partial/\partial t$ and $\vec{p} = -i\hbar\nabla$ which we are used to from Quantum Mechanics. This is connected to the reversed $\phi(\Lambda x)$ at the end of Lecture 5.