Lecture 5: Sept. 19, 2013

First Applications of Noether's Theorem

Now it is time to use the very powerful though abstract formalism Noether developed for continuous symmetries to ask about symmetries we expect our theories to have. At the very least, in this class, we are going to deal only with theories which are invariant under

- spatial translations, $\vec{x} \to \vec{x}\,' = \vec{x} + \vec{c}$.
- time translations, $t \to t' = t + c^0$, or in four dimensional notation, $x^0 \to x'^0 = x^0 + c^0$.
- rotations, $x^i \to x'^i = \sum_j R^i{}_j x^j$, with $R^i{}_j$ an orthogonal matrix of determinant 1.
- Lorentz boost transformations.

We will also consider internal symmetries.

The first two of these transformations together are four dimensional translations,

$$x^\mu \to x'^\mu = x^\mu + c^\mu, \tag{1}$$

and the last two (actually Lorentz transformations already include both) can be written $x^\mu \to x'^\mu = \sum_\nu \Lambda^\mu{}_\nu x^\nu = \Lambda^\mu{}_\nu x^\nu$, (using the Einstein summation convention), where the matrix Λ is a real matrix satisfying the pseudo-orthogonality condition

$$\Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\tau = g_{\nu\tau},$$

which is required so that the length of a four-vector is preserved, $x'^2 := x'^\mu x'_\mu = x^2$.

All together, this symmetry group is called the inhomogeneous Lorentz group, or Poincaré group.

1 Translation Invariance

First let us consider translation invariance with Eq. (1) with $\delta x^\nu = \epsilon c^\nu$. We have here four different generators, with an index ν, each of which has

a conserved four-vector current which is known as the energy-momentum tensor $T^\nu{}_\mu$, with $J^\mu \to c^\nu T^\mu{}_\nu$. We expect all fields to transform as scalars under translations, so $\phi'(x') = \phi(x)$, $\delta\phi = 0$. For homework you found, for Klein-Gordon fields, that

$$T^{\mu\nu} = (\partial^\mu \phi_i)\partial^\nu \phi_i - \mathcal{L} g^{\mu\nu} = (\partial^\mu \phi_i)\partial^\nu \phi_i - \frac{1}{2} g^{\mu\nu}\left((\partial^\rho \phi_i)\partial_\rho \phi_i - m^2\phi^2\right).$$

The conserved charge for translations is the total 4-momentum

$$P^\nu(t) = \int d^3x\, T^{0\nu}(\vec{x}, t),$$

whose zeroth component is, of course the energy or Hamiltonian, and indeed

$$H = P^0 = \int d^3x\, T^{00}(\vec{x}, t) = \int d^3x\, \left(\dot\phi^2 - \mathcal{L}\right) = \int d^3x\, \mathcal{H},$$

which encourages us to think, correctly, that $T^{00} = \mathcal{H}$ in general, and indeed

$$\dot\phi^2 - \mathcal{L} = \dot\phi^2 - \frac{1}{2}\left[\dot\phi^2 - (\nabla\phi)^2 - m^2\phi^2\right] = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 = \mathcal{H}.$$

From the general expression for P^ν in terms of $T^{0\nu}$ and the expression for the latter in the free Klein Gordon theory, we see that

$$\vec{P}(t) = P^j(t) = \int d^3x T^{0j}(\vec{x}, t) = -\int d^3x\, \dot\phi(\vec{x}, t)\vec\nabla\phi(\vec{x}, t),$$

which verifies the expression we used earlier.

Before we go on to consider our next symmetry, I want to say a few words about $T^{\mu\nu}$. This tensor density is called the *energy-momentum tensor* or the *stress-energy tensor*. As we have seen, T^{00} is the energy density and T^{0j} is the density of the j'th component of momentum. But $T^{j\mu}$ also has an interpretation, as the flux through a surface perpendicular to the j direction. T^{j0} is the flux of energy, and T^{ji} is the flux of i'th component of momentum.

The energy momentum tensor plays a crucial role in general relativity, where Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu},$$

gives the coupling of the curvature of space-time, of which $R_{\mu\nu}$ is a piece, to the energy-momentum $T_{\mu\nu}$ of matter (including photons). In this equation $g_{\mu\nu}$ is not the fixed matrix we are using to describe flat space in the absence of gravity, but rather dynamical degrees of freedom, and $R = g^{\mu\nu}R_{\mu\nu}$. This equation shows that in relativistic gravity, it is the energy-momentum tensor, and not the mass, that determines the coupling of matter to gravity.

2 Internal Symmetries

Let us again consider that we might have more than one scalar field, with a Lagrangian density $\sum_i(\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m_i^2\phi_i^2)$. In our noninteracting theory there is no coupling between these, so the expression for the canonical momentum and the creation and annihilation operators apply for each i, and the expressions for $\mathcal{H}$, $\vec{P}$ and $T^{\mu\nu}$ are all just sums over i of what we have derived before. One new issue is the possibility of an internal symmetry. If the masses m_i are all the same (or if we consider a subset of the i for which they are the same), the sum over i becomes a dot product in an abstract vector space, with dimension N given by the number of independent ϕ fields. Then we would expect that there is a symmetry under rotations: $\phi_i(\vec{x},t) \to \phi_i'(\vec{x},t) = \mathcal{O}_{ij}\phi_j(\vec{x},t)$, where $\mathcal{O}$ is an orthogonal $N \times N$ matrix. Here our symmetry involves no variation in the coordinates x^μ, and so we are talking about an *internal symmetry*. The condition for orthogonality is that the elements are real and that the length of dot products is unchanged, $A \cdot B = A_iB_i = (\mathcal{O}A)\cdot(\mathcal{O}B) = \mathcal{O}_{ij}A_j\mathcal{O}_{ik}B_k$, for all A and B, which requires $\mathcal{O}_{ij}\mathcal{O}_{ik} = \delta_{jk}$, or $\mathcal{O}^T\mathcal{O} = 1\!\!1$.

Orthogonal matrices in N dimensions form a group[1] called $O(N)$. From the determinant of $\mathcal{O}^T\mathcal{O} = 1\!\!1$ we see that $\det\mathcal{O} = \pm 1$. The subset of $O(N)$ with $\det\mathcal{O} = 1$ is called $SO(N)$ and forms a continuous, or Lie, group, with $N(N-1)/2$ generators L_ℓ which are antisymmetric real matrices. Thus each element[2] g of $SO(N)$ can thus be written as[3] $g = e^{a_\ell L_\ell}$ for some set of $N(N-1)/2$ real numbers a_ℓ.

For each ℓ there is a Noether current J^μ_ℓ. For homework you found the expressions for these currents and their conserved charges.

What does the existence of a symmetry tell us? If Λ is a symmetry transformation and $|\psi\rangle$ an arbitrary state of the system, the transformation will take us to another state $U(\Lambda)|\psi\rangle$

$$|\psi\rangle \underset{\Lambda}{\longrightarrow} U(\Lambda)|\psi\rangle .$$

That it goes into some state is certain, with probability 1, so $U(\Lambda)$ must preserve the normalization, and $U(\Lambda)$ is therefore a unitary operator on the set of all possible states of the system. If $\mathbf{O}$ is some operator, perhaps $\phi(x)$ or $\pi(x)$ or $\mathcal{H}(x)$, that can act on the states of the system, the state it forms acting on $|\psi\rangle$ is also transformed under the symmetry

$$\mathbf{O}|\psi\rangle \underset{\Lambda}{\longrightarrow} U(\Lambda)\mathbf{O}|\psi\rangle = U(\Lambda)\mathbf{O}U^{-1}(\Lambda)U(\Lambda)|\psi\rangle ,$$

so we can see that the effect of the symmetry on the operator itself is

$$\mathbf{O} \underset{\Lambda}{\longrightarrow} U(\Lambda)\mathbf{O}U^{-1}(\Lambda).$$

In classical physics, if $\phi(\vec{x},t)$ is a solution of the equations of motion, and $\Lambda : \phi \mapsto \phi'$ is a symmetry, then ϕ' will also be a solution of the equations of motion. Quantum mechanically, the transition amplitude $\langle\psi_2,t_2|\psi_1,t_1\rangle$ must be the same as that between the transformed states[4] $\langle U(\Lambda)\psi_2,t_2|U(\Lambda)\psi_1,t_1\rangle$. In particular, states are transformed into states with the same energy. In our theory here, but also for sufficiently unperturbed interacting theories, that means that single particle states are transformed into other single particle states. Thus the set of single particle states form a representation[5].

The theory of representations of groups is a well developed subject, done by mathematicians but studied principally by physicists in courses such as 618. The states which correspond to our fields ϕ_i transform according to the *fundamental* N-dimensional representation. For a noninteracting theory this doesn't teach us much that isn't already obvious, but we can imagine

[1]A group G is a set of elements $g_1, g_2, \ldots$ together with a "multiplication rule", $\circ$, with the properties that 1) the group is closed under multiplication ($g_1 \circ g_2 \in G$); 2) $\circ$ is associative: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$; 3) there is an identity element $1\!\!1$ with $g \circ 1\!\!1 = g$ for all $g \in G$; 4) for every $g \in G$ there is an inverse element g^{-1} with $g \circ g^{-1} = 1\!\!1$. For physicists, the elements are (almost?) always transformations and the multiplication is composition, that is, sequential application. The identity element is the "do-nothing" transformation. As should be familiar from the rotation group, symmetry transformations do not always commute, so the multiplication rule is not necessarily commutative, $g_1 \circ g_2$ may not equal $g_2 \circ g_1$. Composition is automatically associative.

[2]Elements of $O(N)$ which have determinant -1 cannot be reached by a continuous path starting from the identity, which has determinant $+1$, and so cannot be written in this fashion.

[3]Summation over ℓ understood, of course. Physicists usually write $g = e^{-ia_\ell L_\ell}$ with hermitean generators L_ℓ, which for $O(N)$ would be imaginary.

[4]I will only consider transformations which leave time invariant in this statement. Also, I am being careless about phases here.

[5]To a mathematician, a representation is a map from the group into the set of $N \times N$ matrices, $g \mapsto M(g)$, with the property that for all $g, g' \in G$, $M(g \circ g') = M(g)M(g')$. Physicists usually call the N dimensional vector space on which these matrices act the representation, but mathematicians rarely talk about that space, and when they do, they call it a module.

that when we add interactions there may be a coupling to other particles or bound states of these particles. Whatever states there are, they will have to transform according to some representation of the symmetry group[6]. Thus, for example, we cannot have a multiplet of two real scalars if the symmetry group is $SO(3)$.

3 Representations

It is easy to get confused by the duality between active and passive views of symmetries, and between representations and their adjoints, so let's spell out things more carefully. Suppose we have a group G of symmetry transformations and a set of quantum states closed under the action of these symmetries, which will act as linear operators. Let $|\psi_i\rangle$ be a basis (orthonormal) for these states. Under the action of a symmetry g, each state is transformed into a linear combination of these basis elements, so for each i,

$$U(g)\,|\psi_i\rangle = \sum_j M_{ji}(g)\,|\psi_j\rangle\,.$$

You may ask why we have written the matrix with the sum on the first index. First, this is required for the product of two transformations to work correctly:

$$\begin{aligned} U(g_1)U(g_2)\,|\phi_i\rangle &= U(g_1)\left(\sum_j M_{ji}(g_2)\,|\phi_j\rangle\right) \\ &= \sum_j M_{ji}(g_2)U(g_1)\,|\phi_j\rangle \\ &= \sum_j M_{ji}(g_2)\sum_k M_{kj}(g_1)\,|\phi_k\rangle \\ &= U(g_1g_2)\,|\phi_i\rangle = \sum_k M_{ki}(g_1g_2)\,|\phi_k\rangle\,, \end{aligned}$$

where on the second line we noted that the matrix elements M are just c-numbers and so commute with all operators. The last line requires only

[6]Unless there is some form of symmetry breaking, the symmetry group is the group of symmetries of the action, as we discussed above. There are, however, theories with spontaneous symmetry breakdown, which we will discuss later on. In this case, it is only representations of the unbroken subgroup that describe the states of the system.

that

$$M_{ki}(g_1g_2) = \sum_j M_{kj}(g_1)M_{ji}(g_2) \quad \text{or} \quad M(g_1g_2) = M(g_1)M(g_2),$$

which is the definition of a representation.

Second, an arbitrary state in the subspace $|\psi\rangle = \sum_j a_j\,|\psi_j\rangle$ then gets transformed by

$$|\psi\rangle \to U(g)\,|\psi\rangle = \sum_j a_j \sum_k M_{kj}\,|\psi_k\rangle = \sum_k a'_k\,|\psi_k\rangle\,,$$

so we have for the coefficients

$$a \to a' \quad \text{with } a'_k = \sum_j M_{kj}a_j,$$

and the matrix acts on the coefficients as you might expect, with the sum on the second index.

To repeat: A map from a group G into the set of $N \times N$ matrices $M : g \mapsto M(g)$ which has the property that

$$\text{for all } g_1 \text{ and } g_2 \text{ in } G,\ M(g_1g_2) = M(g_1)M(g_2)$$

is called a *representation* of G. Physicists generally use that word to describe the vector space acted on by the matrices, but mathematicians mean the matrices themselves.

A representation is *unitary* if every group element is represented by a unitary matrix. A group which is *compact*[7] has all its finite dimensional irreducible[8] representations equivalent to a unitary one, and that is what we will always deal with. But for a noncompact *semi-simple*[9] group, there are no finite-dimensional unitary *faithful*[10] representations. For an *Abelian*

[7]Roughly, this means that the parameter space of the group is a compact set, as for example the set of rotations in three dimensions, which can be specified by a rotation vector of length at most π. The set of translations or of Lorentz transformations, however, is not compact.

[8]A representation is *reducible* if there is a proper subspace of the module which is mapped only into itself under the actions of all elements of the group. For the groups and representations we will consider, all representations can be written as a direct sum of irreducible representations.

[9]A group is semi-simple if there is no Abelian subgroup $\mathcal{A}$ which is invariant. A subgroup $\mathcal{A}$ is invariant if $gag^{-1} \in \mathcal{A}$ for all $g \in G$ and $a \in \mathcal{A}$

[10]$g \mapsto 1$ only for $g = 1$.

group, for which all of the elements commute with each other, all irreducible representations are one-dimensional. The only non-compact groups we will discuss are the translation and Lorentz/Poincaré groups, while compact internal symmetry groups, especially SU(3), will play a big role.

For the translations, of course, the group is Abelian, so the irreducible representations of the translation group are each a single state $|p^\mu, \alpha\rangle$, where α represents all other properties of the state, which are necessarily invariant under translations. Then under a translation $x^\mu \to x^\mu + c^\mu$ the state is multiplied by a phase

$$|p^\mu, \alpha\rangle \to e^{-ic^\mu P_\mu} |p^\mu, \alpha\rangle = e^{-ic^\mu p_\mu} |p^\mu, \alpha\rangle .$$

With $U(T_c) = e^{-ic^\mu P_\mu}$, this says

$$U(T_c) a^\dagger_{\vec{p},\alpha} |0\rangle = e^{-ic^\mu p_\mu} a^\dagger_{\vec{p},\alpha} |0\rangle = e^{-ic^\mu p_\mu} a^\dagger_{\vec{p},\alpha} U(T_c) |0\rangle ,$$

which suggests

$$U(T_c) a^\dagger_{\vec{p},\alpha} U^{-1}(T_c) = e^{-ic^\mu p_\mu} a^\dagger_{\vec{p},\alpha},$$

and differentiating with respect to c^μ at $c = 0$ gives

$$i[P_\mu, a^\dagger_{\vec{p},\alpha}] = i p_\mu a^\dagger_{\vec{p},\alpha},$$

which we have previously verified for the scalar field.

These momentum eigenstates are not individually representations of the Poincaré group, however, because rotations or Lorentz transformations convert a state of one momentum p^μ into a state with another. Thus we expect that under a Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\mu$ we should have

$$|p^\mu, a, \alpha\rangle \to U(\Lambda) |p^\mu, a, \alpha\rangle = \sum_{p', b} M_{(p',b),(p,a)}(\Lambda) |p'^\mu, b, \alpha\rangle ,$$

where I have divided the extra properties into those left invariant under Lorentz transformations (α) and those which get mixed, (a, b), like the L_z quantum number m for rotations of a particle at rest. But in fact only one p' should enter for a fixed Lorentz transformation on a fixed p^μ, namely $p'^\mu = \Lambda^\mu{}_\nu p^\nu$. So the matrix M need have only the discrete indices (a, b), but it can depend on the momentum of the state it acts on:

$$|p^\mu, a, \alpha\rangle \to U(\Lambda) |p^\mu, a, \alpha\rangle = \sum_b M_{b,a}(\Lambda, p^\mu) \left|\Lambda^\mu{}_\nu p^\nu, b, \alpha\right\rangle .$$

This is rather messy, so let's look first at scalar particles, for which the matrix M is just 1. Let's also suppress the α index Then

$$U(\Lambda) |p^\mu\rangle = \left|\Lambda^\mu{}_\nu p^\nu\right\rangle = |p'^\mu\rangle$$

which means that, at least acting on the vacuum,

$$U(\Lambda)\sqrt{2E_p}\, a^\dagger_{\vec{p}} U^{-1}(\Lambda) = \sqrt{2\Lambda^0{}_\nu p^\nu}\, a^\dagger_{\Lambda^j{}_\nu p^\nu}.$$

Taking the hermitean conjugate of this,

$$U(\Lambda)\sqrt{2E_p}\, a_{\vec{p}} U^{-1}(\Lambda) = \sqrt{2\Lambda^0{}_\nu p^\nu}\, a_{\Lambda^j{}_\nu p^\nu}.$$

and thus for a scalar field[11]

$$\phi(x^\mu) = \int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2)\Theta(p^0)\sqrt{2p^0}\left(a_{\vec{p}}\, e^{-ip_\mu x^\mu} + a^\dagger_{\vec{p}}\, e^{ip_\mu x^\mu}\right),$$

$$\begin{aligned} U(\Lambda) \quad \phi(x^\mu) \quad U^{-1}(\Lambda) &= \int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2)\Theta(p_0)U(\Lambda)\sqrt{2E_p} \\ &\qquad \left(a_{\vec{p}}\, e^{-ip_\mu x^\mu} + a^\dagger_{\vec{p}}\, e^{ip_\mu x^\mu}\right) U^{-1}(\Lambda) \\ &= \int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2)\Theta(p_0)\sqrt{2E_{p'}}\left(a_{\vec{p}'} e^{-ip_\mu x^\mu} + a^\dagger_{\vec{p}'} e^{ip_\mu x^\mu}\right) \\ &= \int \frac{d^4p'}{(2\pi)^3}\delta(p'^2 - m^2)\Theta(p'_0)\sqrt{2E_{p'}}\left(a_{\vec{p}'} e^{-ip'_\mu x'^\mu} + a^\dagger_{\vec{p}'} e^{ip'_\mu x'^\mu}\right) \\ &= \phi(x'^\mu), \end{aligned}$$

where I have used the Lorentz invariance of the integration measure $\int d^4p\, \delta(p^2 - m^2)\Theta(p_0)$, defined $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, and used the Lorentz invariance to replace $e^{ip_\mu x^\mu} = e^{ip'_\mu x'^\mu}$. Thus we see that for a scalar field

$$U(\Lambda)\phi(x^\mu)U^{-1}(\Lambda) = \phi(\Lambda^\mu{}_\nu x^\nu).$$

This is a bit surprizing, and looks backwards from our classical statement $\phi \to \phi'$ where $\phi'(\Lambda x) = \phi(x)$, or $\phi'(x) = \phi(\Lambda^{-1}(x))$. An explanation, sort of, is on pages 59-60 of the book, pointing out that earlier we were considering

[11]Combining PS 2.25 and 2.40.

a fixed field configuration and defining a new field in new coordinates with the same physics as the old field. This is a passive view of what a Lorentz transformation means. Here we are taking an operator that creates a particle at one point and rotating it so that it creates a particle at a transformed point, an active view of the Lorentz transformation. These are effectively inverses of each other. It is essential not to think that $U(\Lambda)\phi(x)U^{-1}(\Lambda)$ is $\phi'(x)$ or $\phi'(\Lambda x)$, neither of which is true.