# Noether's Theorem, (condensed version) Sept. 16, 2013



Instead of going through the detailed version of my Lecture 4: Noether's Theorem, which pretty closely follows Goldstein, I will just outline the argument. But you should go through the full version at some point in your career. See `lect04`.

We have long known that if a coordinate does not appear undifferentiated in the Lagrangian, it is an "ignorable" coordinate, and the conjugate momentum is conserved. If $L(\vec{r}, \dot{\vec{r}})$ does not depend on one coordinate $x$, $p_x = \partial L/\partial \dot{x}$ is conserved, and if $L(r, \theta, \phi, \cdots)$ does not depend on $\phi$, $p_\phi = L_z$ is conserved. But, if the potential depends only on $r$, so are $L_x$ and $L_y$, though these are not conjugate to an ignorable coordinate, and $\theta$ is not ignorable.

An ignorable coordinate $q_i$ corresponds to a symmetry group $q_i \to q_i + c$, where $c$ is an arbitrary constant. We can view this group of symmetries as generated by and infinitesimal transformation $q_i \to q_i + \delta q_i$. But there can be more complicated symmetries of the Lagrangian, for example $\vec{r} \to \vec{r} + \delta\vec{\omega} \times \vec{r}$, which is a rotation about $\delta\vec{\omega}$, and the three generators generate the rotation group and insures the conservation of all three components of the angular momentum.

For fields, the possible symmetries may be more complicated yet, for the value of each point in space may be changed by the symmetry, possibly in a way that depends on fields at other points.

The simplest situation is for an "internal symmetry", where the change of $\phi_i(x^\mu)$ depends only on fields at $x^\mu$,

$$\phi_i(x^\mu) \to \phi_i(x^\mu) + \delta\phi_i\left(x^\mu; \phi_j(x^\mu)\right).$$

To be very general, however, we might consider that $\delta\phi_i(x^\mu)$ could depend on $\phi_j(y^\mu)$ everywhere. This is, however, much too general, but we do want to consider the possibility that the symmetry changes the $x^\mu$ coordinates, relating the fields at some new point $x'^\mu$ to those at an old point $x^\mu$, as for example what happens under rotation, where $\vec{E}'(\vec{x}')$ is a vector rotated from $\vec{E}(\vec{x})$, not from $\vec{E}(\vec{x}')$.

For an infinitesimal symmetry transformation, the new point $x'^\mu$ will differ by an infinitesimal from $x^\mu$,

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu.$$

The simplest field transforms like a scalar:

$$\phi'(x') = \phi(x) \qquad \text{for a scalar field}$$

but more generally

$$\phi'_j(x') = \phi_j(x) + \delta\phi_j\left(x, \phi_k(x)\right),$$

which is what you would need if $\phi$ were a vector, with $\delta\vec{\phi} = \delta\vec{\omega} \times \vec{\phi}$.

Note that I have defined $\delta\phi_j = \phi'_j(x') - \phi_j(x)$, which is not the same as the change of $\phi$ at a given argument,

$$\mathfrak{d}\phi_j(x) = \phi'_j(x) - \phi_j(x).$$

Now if we have a symmetry, that means that the new fields $\phi'_j(x')$ should have the same *action* as the old fields $\phi_j(x)$ with the action defined by the integral over the equivalent space-time region $R'$, which is the image of the original region $R$ in the $x$ coordinates, and with the fixed values at the boundary specified on the new boundary $\delta R'$. As a consequence, we expect the Lagrangian *density* to transform like a scalar density, not like a scalar,

$$\delta\mathcal{L} := \mathcal{L}'(\phi'_j; \partial'_\mu\phi'_j, x'_\mu) - \mathcal{L}(\phi_j; \partial_\mu\phi_j, x_\mu)\left|\frac{\partial x^\mu}{\partial x'^{\,\nu}}\right|$$

with a Jacobian, and furthermore, there might be an additional change which would be irrelevant as long as it is a total derivative, $\delta\mathcal{L} = \partial_\mu\Lambda^\mu$.

Under such a symmetry, we can show (see the full version of Lecture 4)

$$J^\mu \propto -\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_j}\delta\phi_j + \frac{\mathcal{L}}{\partial\partial_\mu\phi_j}\left(\partial_\nu\phi_j\right)\delta x^\nu - \mathcal{L}\delta x^\mu + \Lambda^\mu$$

is a conserved current, with $\partial_\mu J^\mu = 0$ for fields obeying the equations of motion, and

$$Q(t) := \int d^3x J^\mu(\vec{x}, t)$$

is a conserved charge (with provisos we ignore for now.).