# Lecture 4: Noether's Theorem Sept. 16, 2013



I am sure you have heard that for every continuous transformation of the coordinates that one can make without affecting the Lagrangian, there is a conserved quantity.

The simplest example is an ignorable coordinate, $q_j$, which does not appear in the Lagrangian. An example of this is the azimuthal coordinate $\phi$ for a particle in a spherically symmetric potential, or the horizontal position $x$ in ballistic motion. The conserved quantities are then $L_z$ and $p_x$ respectively.

Sometimes each of the $q_i$ does appear, but in a way that some combination can change without affecting the Lagrangian. For example, an isolated system of particles at $\vec{r}_j$, interacting with each other with potentials that depend only on separations $\vec{r}_i - \vec{r}_j$, is invariant under all the $\vec{r}_j$ being changed by the same amount, $\vec{r}_j \to \vec{r}_j + \vec{b}$, corresponding to translating the whole system by $\vec{b}$. Then the *total* momentum $\vec{P} = \sum_i \vec{p}_i$ is conserved.

In field theory the dynamical variables $\eta$ are indexed by continuous parameters $\vec{r}$, which complicates the form of symmetry transformations that $\eta(\vec{r})$ can have. But Noether's theorem provides us with a framework for discussing the general case, which will prove to have many applications in quantum field theory.

## 1 Noether's Theorem

Now I want to give a thorough discussion of Noether's theorem,[1] which relates continuous symmetries of a theory to conserved currents and conserved charges, for classical fields. The treatment in Peskin and Schroeder is not very intuitive for transformations in which the coordinates change. When we consider transformations such as translations or rotations, we expect the physics to be invariant because we view such changes as passive, as changes in our coordinate system describing what is really an unchanged physical situation, though we can also consider the transformation in an active sense, that the physics would be unchanged if we actually picked up the whole universe and moved or rotated it some fixed amount before proceeding with the experiments. For transformations such as rotations or Lorentz transformations,

[1] This section relies heavily on Goldstein, "Classical Mechanics", 2nd Ed., section 12-7.

the change in fields, $\phi(x)$, viewed in an active sense, will have an explicit $x$ dependence, but in the passive view we see that this is an expected feature of the change of the coordinate at which the field is evaluated. The transformation is still a global one, described by parameters that are independent of $x$. We will nonetheless permit coordinate-dependent field transformations in our treatment, in general because they help define the current, but also as actual symmetry transformations which we will meet later when we discuss local gauge transformations. For Lorentz transformations, including rotations, the symmetry comes about becaue the same physics is described as being at different coordinates, with $\phi'(x') \sim \phi(x)$. While symmetries involving infinitesimal variations in the coordinates can also be considered as local changes in the fields at the same value of $x$, as Peskin and Schroeder do, it is much more intuitive to treat these as relating new fields at new values of $x$ to old fields at old values of $x$. But the development of the framework for this more general transformation will be, I'm afraid, a bit formal.

Symmetries of a Lagrangian and conserved quantities are intimately related. For example, we know that the momentum conjugate to an ignorable coordinate is conserved. The general connection is due to the famous theorem of Emmy Noether. We will consider infinitesimal tranformations of the field degrees of freedom $\phi_i(x^\mu)$ which relate the new value of the field to the old value at some other value of the coordinates, one which we consider to describe, in the old coordinates, the same physical point that the new $x'$ describes in the new coordinates. That is, the new fields $\phi'_i(x')$ are related to $\phi_j(x)$ rather than $\phi_j(x')$, where

$$x_\mu \to x'_\mu = x_\mu + \delta x_\mu \tag{1}$$

is some infinitesimal transformation of the coordinates rather than of the degrees of freedom. For a scalar field assumed "invariant" under the transformation, like temperature under a rotation, we would define the new field

$$\phi'(x') = \phi(x).$$

More generally the field may also change, in a way that may depend on other fields,

$$\phi'_i(x') = \phi_i(x) + \delta\phi_i(x; \phi_k(x)). \tag{2}$$

This is what you would expect for a vector field even if it is "physically" unchanged. For example, for the electric field $\vec{E}$ under rotations, the new $E'_x(\vec{r}\,')$ gets a contribution from the old $E_y(\vec{r})$.

To say that

$$x_\mu \to x'_\mu, \qquad \phi_i \to \phi'_i$$

is a symmetry means, at the least, that if $\phi_i(x)$ is a specific solution of the equation of motion, the transformed field $\phi'_i(x')$ is also a solution. The equations of motion are determined by varying the action, so if the corresponding actions are equal for each pair of configurations $(\phi(x), \phi'(x'))$, so are the equations of motion. Notice here that what we are saying is that the same Lagrangian function applied to the fields $\phi'_i$ and integrated over $x' \in \mathcal{R}'$ should give the same action as $S = \int_{\mathcal{R}} \mathcal{L}(\phi_i(x)...)d^4x$, where $\mathcal{R}'$ is the range of $x'$ corresponding to the domain $\mathcal{R}$ of $x$.
[Of course our argument applies also if $\delta x_\mu = 0$, that the transformation does not involve a change in coordinates. Such symmetries are called *internal symmetries*, with isospin an example.]

Actually, the above condition that the actions be unchanged is far more demanding than is needed to insure that the same equations of motion arise. The variations required to derive the equations of motion only compare actions for field configurations unchanged at the boundaries, so if the actions

$$S' = \int_{\mathcal{R}'} \mathcal{L}(\phi'_i(x'), \partial'_\mu \phi'_i(x'), x')d^4x' \quad \text{and} \quad S = \int_{\mathcal{R}} \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x)d^4x \quad (3)$$

differ by a function only of the values of $\phi_i$ on the boundary $\partial\mathcal{R}$, they will give the same equations of motion. Even in quantum mechanics, where the transition amplitude is given by integrating $e^{iS/\hbar}$ over all configurations, a change in the action which depends only on surface values is only a phase change in the amplitude. In classical mechanics we could also have an overall change multiplying the Lagrangian and the action by a constant $c \neq 0$, which would still have extrema for the same values of the fields, but we will not consider such changes because quantum mechanically they correspond to changing Planck's constant.

The Lagrangian density is a given function of the old fields $\mathcal{L}(\phi_i, \partial_\mu \phi_i, x_\mu)$. If we substitute in the values of $\phi(x)$ in terms of $\phi'(x')$ we get a new density $\mathcal{L}'$, defined by

$$\mathcal{L}'(\phi'_i, \partial'_\mu \phi'_i, x'_\mu) = \mathcal{L}(\phi_i, \partial_\mu \phi_i, x_\mu) \left| \frac{\partial x^\nu}{\partial x'^\mu} \right|,$$

where the last factor is the Jacobian of the transformation $x \to x'$, required because these are densities, intended to be integrated. This change in functional form for the Lagrangian is not the symmetry transformation, for as long as $x \leftrightarrow x'$ is one-to-one, the integral is unchanged

$$\begin{aligned}\int_{\mathcal{R}'} \mathcal{L}'(\phi_i'(x'), \partial_\mu' \phi_i'(x'), x') d^4x' &= \int_{\mathcal{R}'} \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x) \left| \frac{\partial x^\nu}{\partial x'^\mu} \right| d^4x' \\ &= \int_{\mathcal{R}} \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x) d^4x = S \qquad (4)\end{aligned}$$

regardless of whether this transformation is a symmetry.

We see that the change in the action, $\delta S = S' - S$, which must vanish up to surface terms for a symmetry, may be written as an integral over $\mathcal{R}'$ of the variation of the Lagrangian density, $\delta S = \int_{\mathcal{R}'} \delta\mathcal{L}$, with

$$\begin{aligned}\delta\mathcal{L}(\phi_i'(x'), \partial_\mu' \phi_i'(x'), x') &:= \mathcal{L}(\phi_i'(x'), \partial_\mu' \phi_i'(x'), x') - \mathcal{L}'(\phi_i'(x'), \partial_\mu' \phi_i'(x'), x') \\ &= \mathcal{L}(\phi_i'(x'), \partial_\mu' \phi_i'(x'), x') - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x) \left| \frac{\partial x^\nu}{\partial x'^\mu} \right|. \qquad (5)\end{aligned}$$

Here we have used the first of Eq. (3) for $S'$ and Eq. (4) for $S$.

Expanding to first order, the Jacobian is

$$\det \left| \frac{\partial x'^\mu}{\partial x^\nu} \right|^{-1} = \det \left(\delta^\mu_\nu + \partial_\nu \delta x^\mu\right)^{-1} = \left(1 + \operatorname{Tr} \frac{\partial \delta x^\mu}{\partial x^\nu}\right)^{-1} = 1 - \partial_\mu \delta x^\mu, \qquad (6)$$

while

$$\begin{aligned}\mathcal{L}(\phi_i'(x'), \partial_\mu' \phi_i'(x'), x') &= \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x) \\ &\quad + \delta\phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + \delta(\partial_\mu \phi_i) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} + \delta x^\mu \frac{\delta \mathcal{L}}{\delta x^\mu}, \qquad (7)\end{aligned}$$

Thus[2]

$$\delta\mathcal{L} = \mathcal{L} \partial_\mu \delta x^\mu + \delta\phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + \delta(\partial_\mu \phi_i) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} + \delta x^\mu \frac{\delta \mathcal{L}}{\delta x^\mu}, \qquad (8)$$

and if this is a divergence, $\delta\mathcal{L} = \partial_\mu \Lambda^\mu$, we will have a symmetry.

There are subtleties in this expression[3]. The last term involves a derivative of $\mathcal{L}$ with its first two arguments fixed, and as such is not the derivative with respect to $x^\mu$ with the *functions* $\phi_i$ fixed. For this reason we used a

[2] This is the equation to use on homework.

[3] There is also a summation understood on the repeated $i$ index as well as on the repeated $\mu$ index.

different symbol, because it is customary to use $\partial_\mu$ to mean only that $x^\nu$ is fixed for $\nu \neq \mu$, and not to indicate that the other arguments of $\mathcal{L}$ are held fixed. That form of derivative is the stream derivative,

$$\frac{\partial\mathcal{L}\Big(\phi_i(x), \partial_\mu\phi_i(x), x\Big)}{\partial x^\nu} = \frac{\delta\mathcal{L}\Big(\phi_i(x), \partial_\mu\phi_i(x), x\Big)}{\delta x^\nu} + (\partial_\nu\phi_i)\frac{\partial\mathcal{L}}{\partial\phi_i} + (\partial_\nu\partial_\mu\phi_i)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}.$$

Note also that $\delta\phi_i(x) = \phi'_i(x') - \phi_i(x)$ is not simply the variation of the field at a point, $\mathfrak{d}\phi_i(x) = \phi'_i(x) - \phi_i(x)$, but includes in addition the change $(\delta x^\mu)\partial_\mu\phi_i$ due to the displacement of the argument (1). Thus

$$\delta\phi_i(x) = \mathfrak{d}\phi_i(x) + (\delta x^\nu)\partial_\nu\phi_i. \tag{9}$$

The variation with respect to $\partial'_\mu\phi'_i$ needs to be examined carefully, because the $\delta$ variation effects the coordinates, and therefore in general $\partial_\mu\delta\phi_i \neq \delta\partial_\mu\phi_i$. By definition,

$$\begin{aligned}
\delta\partial_\mu\phi_i &= \partial\phi'_i/\partial x'^\mu\big|_{x'} - \partial\phi_i/\partial x^\mu\big|_x \\
&= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\left[\phi_i + (\delta x^\rho)\partial_\rho\phi_i + \mathfrak{d}\phi_i\right]\Big|_x - \partial\phi_i/\partial x^\mu\big|_x \\
&= -\left(\partial_\mu\delta x^\nu\right)\partial_\nu\phi_i + \frac{\partial}{\partial x^\mu}\left[(\delta x^\rho)\partial_\rho\phi_i + \mathfrak{d}\phi_i\right] \\
&= (\delta x^\nu)\partial_\mu\partial_\nu\phi_i + \mathfrak{d}\partial_\mu\phi_i
\end{aligned} \tag{10}$$

where in the last line we used $\partial_\mu\mathfrak{d}\phi_i = \mathfrak{d}\partial_\mu\phi_i$, because the $\mathfrak{d}$ variation *is* defined at a given point and *does* commute with $\partial_\mu$. Alternatively, we might have rewritten the third line as

$$\delta\partial_\mu\phi_i = \frac{\partial}{\partial x^\mu}\delta\phi_i - \left(\partial_\mu\delta x^\nu\right)\partial_\nu\phi_i.$$

Notice that the $\delta x^\nu$ terms in (9) and (10) are precisely what is required in (7) to change the last term to a full stream derivative. Thus

$$\begin{aligned}
\mathcal{L}(\phi'_i(x'), \partial'_\mu\phi'_i(x'), x') &= \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x), x) \\
&\quad + \mathfrak{d}\phi_i\frac{\partial\mathcal{L}}{\partial\phi_i} + \mathfrak{d}\partial_\mu\phi_i\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} + \delta x^\mu\frac{\partial\mathcal{L}}{\partial x^\mu},
\end{aligned} \tag{11}$$

where now $\partial\mathcal{L}/\partial x^\mu$ means the stream derivative, including the variations of $\phi_i(x)$ and its derivative due to the variation $\delta x^\mu$ in their arguments.

Inserting this and (6) into the expression (5) for $\delta\mathcal{L}$, we see that the change of action is given by the integral of

$$\begin{aligned}\delta\mathcal{L} &= (\partial_\mu \delta x^\mu)\,\mathcal{L} + \delta x^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} + \mathfrak{d}\phi_i \frac{\partial \mathcal{L}}{\partial \phi_i} + \mathfrak{d}\partial_\mu\phi_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \\ &= \frac{\partial}{\partial x^\mu}\left(\delta x^\mu \mathcal{L} + \mathfrak{d}\phi_i \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i}\right) + \mathfrak{d}\phi_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i}\right) \qquad (12)\end{aligned}$$

We will discuss the significance of this in a minute, but first, I want to present an alternate derivation.

The alternative derivation of this result is based on the observation that in the expression (3) for $S'$, $x'$ is a dummy variable and can be replaced by $x$, and the difference can be taken at the same $x$ values, except that the ranges of integration differ. Thus

$$S' = \int_{\mathcal{R}'} \mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x), x\right) d^4x.$$

This differs from $S(\phi)$ because

1. the Lagrangian is evaluated with the field $\phi'$ rather than $\phi$, producing a change

$$\delta_1 S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi_i}\mathfrak{d}\phi_i + \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i}\mathfrak{d}\partial_\mu\phi_i\right) d^4x,$$

   where the variation with respect to the fields is now in terms of $\mathfrak{d}\phi_i(x) := \phi_i'(x) - \phi_i(x)$, at the same argument $x$.

2. Change in the region of integration, $\mathcal{R}'$ rather than $\mathcal{R}$,

$$\delta_2 S = \left(\int_{\mathcal{R}'} - \int_{\mathcal{R}}\right) \mathcal{L}(\phi_i, \partial_\mu\phi_i, x)\, d^4x.$$

If we define $dS_\mu$ to be an element of the three dimensional surface $\partial\mathcal{R}$ of $\mathcal{R}$, with outward-pointing normal in the direction of $dS_\mu$, the difference in the regions of integration may be written as an integral over the surface,

$$\left(\int_{\mathcal{R}'} - \int_{\mathcal{R}}\right) d^4x = \int_{\partial\mathcal{R}} \delta x^\mu \cdot dS_\mu.$$

Thus

$$\delta_2 S = \int_{\partial\mathcal{R}} \mathcal{L}\delta x^\mu \cdot dS_\mu = \int_{\mathcal{R}} \partial_\mu \left(\mathcal{L}\delta x^\mu\right) \qquad (13)$$

by Gauss' Law (in four dimensions).

As $\mathfrak{d}$ is a difference of two functions at the same values of $x$, this operator commutes with partial differentiation, so $\mathfrak{d}\partial_\mu\phi_i = \partial_\mu\mathfrak{d}\phi_i$. Using this in the second term of $\delta_1 S$

$$\delta_1 S = \int_{\mathcal{R}} d^4x \left[\partial_\mu\left(\mathfrak{d}\phi_i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\right) + \mathfrak{d}\phi_i\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\right)\right]$$

Thus altogether $S' - S = \delta_1 S + \delta_2 S = \int_{\mathcal{R}} d^4x\, \delta\mathcal{L}$, with $\delta\mathcal{L}$ given by (12). This completes our alternate derivation that $S' - S = \int_{\mathcal{R}} d^4x\delta\mathcal{L}$, and Eq. (12).

Note that $\delta\mathcal{L}$ is a divergence plus a piece which vanishes if the dynamical fields obey the equation of motion, quite independent of whether or not the infinitesimal variation we are considering is a symmetry. As we mentioned, to be a symmetry, $\delta\mathcal{L}$ must be a divergence for all field configurations, so that the variations over configurations will give the correct equations of motion (classically) or, for the functional integral formulation of quantum mechanics, so that all the paths will contribute equivalently.

We have been assuming the variations $\delta x$ and $\delta\phi$ can be treated as infinitesimals. This is appropriate for a continuous symmetry, that is, a symmetry group[4] described by a (or several) continuous parameters. For example, symmetry under displacements $x^\mu \to x^\mu + c^\mu$, where $c^\mu$ is any arbitrary fixed 4-vector, or rotations through an arbitrary angle $\theta$ about a fixed axis. Each element of such a group lies in a one-parameter subgroup, and can be obtained, in the limit, from an infinite number of applications of an infinitesimal transformation. If we call the parameter $\epsilon$, the infinitesimal variations in $x^\mu$ and $\phi_i$ are given by derivatives of $x'(\epsilon, x)$ and $\phi'$ with respect to the parameter $\epsilon$. Thus

$$\delta x^\mu = \epsilon\, \frac{dx'^\mu}{d\epsilon}\bigg|_{x^\nu}, \qquad \delta\phi_i = \epsilon\, \frac{d\phi'_i(x')}{d\epsilon}\bigg|_{x^\nu}.$$

The divergence must also be first order in $\epsilon$, so $\delta\mathcal{L} = \epsilon\partial_\mu\Lambda^\mu$ if we have a symmetry.

We define the **current** for the transformation

$$J^\mu = -\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\frac{d\phi'_i}{d\epsilon} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\partial_\nu\phi_i\frac{dx'^\nu}{d\epsilon} - \mathcal{L}\frac{dx'^\mu}{d\epsilon} + \Lambda^\mu. \qquad (14)$$

[4]Symmetries always form a group. Continuous symmetries form a *Lie group*, whose elements can be considered exponentials of linear combinations of generators. The generators form a *Lie algebra*.

Recalling that $\mathfrak{d}\phi_i = \delta\phi_i - (\delta x^\nu)\partial_\nu\phi_i$, we can rewrite (12)

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial}{\partial x^\mu}\left(\delta x^\mu \mathcal{L} + \delta\phi_i \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i} - \delta x^\nu(\partial_\nu\phi_i)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\right) \\ &\quad +\mathfrak{d}\phi_i\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\right)\end{aligned}$$

and see that

$$\begin{aligned}\epsilon\partial_\mu J^\mu &= \frac{\partial}{\partial x^\mu}\left(-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\partial_\nu\phi_i\delta x^\nu\right) - \frac{\partial}{\partial x^\mu}\left(\mathcal{L}\delta x^\mu\right) + \delta\mathcal{L} \\ &= \mathfrak{d}\phi_i\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\right)\end{aligned}$$

Thus we have

$$\partial_\mu J^\mu = 0 \qquad \text{for a symmetry, when the fields obey the equations of motion.}$$

This condition is known as **current conservation**. Associated with each such current, we may define the charge enclosed in a constant volume $V$

$$Q_V(t) = \int_V d^3x J^0(\vec{x}, t).$$

If we evaluate the time derivative of the charge, we have

$$\begin{aligned}\frac{d}{dt}Q_V(t) &= \int_V d^3x\partial_0 J^0(\vec{x},t) \approx -\int_V d^3x \sum_{i=1,3}\partial_i J^i(\vec{x},t) = -\int_V d^3x\vec{\nabla}\cdot\vec{J}(\vec{x},t) \\ &= -\int_{\partial V}\vec{J}\cdot d\vec{S},\end{aligned}$$

where $\partial V$ is the boundary of the volume and $d\vec{S}$ an element of surface area. We have used the conservation of the current and Gauss' Law. If, as can usually be assumed, the current vanishes as we move infinitely far way from the region of interest, the surface integral vanishes if we take $V$ to be all of space, and we find that the total charge is conserved, $dQ/dt = 0$, in the same sense that equations of motion are satisfied. The assumption about asymptotic behavior is not always valid, and we must consider whether we have grounds for it in particular applications. We will see later that in some circumstances there are "anomolies" when this assumption is not justified.

In all of the examples you will consider for homework, using the description of the symmetry transformation described above, we will find $\Lambda^\mu = 0$. These symmetries might have been described differently, however, by not including any change in the coordinates $x^\mu$, considering only the variation of the fields $\phi_i \to \phi_i + \mathfrak{d}\phi_i$. In that treatment the $\delta x^\mu \mathcal{L}$ term would not have appeared explicitly, but would have entered anyway by means of $\Lambda^\mu$.