Gauge Theory on a Lattice

One approach to field theory, in particular to aspects that are not well treated in perturbation theory, is to approximate the field defined on a space-time continuum with a lattice field theory, defined on a lattice of space-time points. There has been a great deal of cross-fertilization from considering renormalization from the continuum high-energy physics point of view and from the condensed matter, Wilsonian, viewpoint, in which the field theory only makes sense down to some small distance scale anyway. But we are going to consider it for a different reason — because it helps to clarify the fundamental idea of gauge fields.

Symmetry

Consider a theory which involves a set of N fields $\phi_i(x^\mu)$ which have an internal symmetry group $\mathcal{G}$ under which they transform with a representation M, so that

$$G : \phi_i(x) \to \phi_i'(x) = \sum_j M_{ij}(G)\phi_j(x). \tag{1}$$

If it is a symmetry, the Lagrangian must be invariant. For the kinetic term $\frac{1}{2}\sum_{\mu,i}\partial_\mu\phi_i\partial^\mu\phi_i$, invariance requires that M is an orthogonal matrix $\sum_k M_{ki}M_{kj} = \delta_{ij}$. That condition also insures the invariance of the mass term $-\frac{1}{2}\sum_i\phi_i^2$, and of any other "potential" term $V(\sum_i\phi_i^2)$ depending only on the "length" of ϕ. Provided V has that form, we see that the theory should be invariant under the orthogonal transformations (1). We see that the individual components ϕ_i are only projections along the unit vectors of an arbitrary orthonormal basis of $\mathbb{R}^N$, and do not have separate intrinsic physical meanings.

Discretization

How might we approximate the continuum theory on a lattice? Instead of $\phi_i(\mathbf{x})$ defined for all values of $\mathbf{x} \in \mathbb{R}^4$, we might have $\phi_i(\vec{n})$ discrete variables defined only for integer values $\vec{n} \in \mathbb{Z}^4$, representing a lattice in space-time with lattice spacing a, with $\mathbf{x}^\mu = an^\mu$. The mass term in the action

$$-\frac{1}{2}\int d^4x \sum_i \phi_i^2(\mathbf{x}) \to -\frac{1}{2}a^4 \sum_{\vec{n}\in\mathbb{Z}^4}\sum_i \phi_i^2(\vec{n}).$$

For the kinetic energy term we need to replace a derivative by a finite difference. The simplest substitution is to replace

$$\partial_\mu\phi_i(\mathbf{x}) \to \frac{1}{a}\left(\phi_i(\vec{n}+\vec{\Delta}_\mu) - \phi_i(\vec{n})\right),$$

where Δ_μ is 1 in the μ direction and 0 in the others. Here the relation of x^μ and $\vec{n}$ is $x^\nu = an^\nu + \frac{1}{2}a\delta^\nu_\mu$, representing most accurately the x in the middle of the two lattice points. If we expand out the squares of the differences, we get terms which look just like the mass terms, but also nearest neighbor couplings $\sum_i \phi_i(\vec{n}+\Delta_\mu)\phi_i(\vec{n})$.

Each of these contributions to the action is still invariant under the transformation (1), providing we use the same group transformation at every point in space-time. This is called a global gauge transformation.

In a relativistic field theory, all information is local, because information can only travel at the speed of light. So we might ask, if the theory is unchanged by a group action at one point, why should that depend on having the same transformation at every other point? In other words, could we have a **local** symmetry, in which equation (1) holds with the group element varying from one point of space-time to another? The mass terms and other terms in $V(\phi)$ only depend on one point, so they don't care whether M varies, and they are invariant under such transformations. But the nearest-neighbor coupling

$$\sum_i \phi_i(\vec{n}+\Delta_\mu)\phi_i(\vec{n}) \to M_{ik}(G(\vec{n}+\Delta_\mu))M_{ij}(G(\vec{n}))\,\phi_k(\vec{n}+\Delta_\mu)\phi_j(\vec{n})$$

is not invariant because

$$M^{-1}(G(\vec{n}+\Delta_\mu))M(G(\vec{n})) \neq 1$$

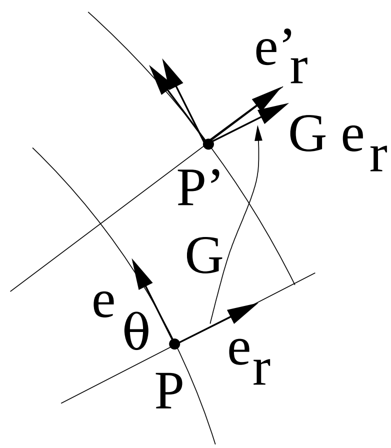
if the G's (and hence the M's) vary from point to point.

Parallel Transport

The problem is that we have a term in the Lagrangian that is a function of how ϕ changes from point to point, but we measure that change by how much the components change. That is only correct if the basis for comparing the ϕ's does not change. We must have a way to measure change from point to point, but before we can subtract one ϕ vector from another at a different point, we must "parallel transport" it to that new point. That is, for each link between neighboring points, we must have a rule for parallel transporting ϕ fields from one end of the link to the other. This introduces new degrees of freedom, which are actually one element of the symmetry group (and therefore perhaps several degrees of freedom, $\frac{1}{2}N(N-1)$ for $SO(N)$, the orthogonal transformations in N dimensions). We can then build a theory with a local symmetry, but at the expense of introducing a lot of new degrees of freedom.

The theory that emerges from these consideration is a **gauge field theory**. Its degrees of freedom include not only the "matter fields" at each site of the lattice, but also "gauge fields" on each link between nearest neighbors. The matter fields live in a vector space which transforms linearly as a representation[1] of the "gauge group" $\mathcal{G}$. The gauge fields live in the group itself, at least in the lattice field theory, but may alternately be considered to take values in the Lie algebra of generators of the group.

What is the meaning that the group element G associated with a given link $(\vec{n}, \vec{n}+\vec{\Delta}_\mu)$? We called it a parallel transport, a definition of how one can transport the basis vectors $\vec{e}_i(\vec{n})$ to the point $\vec{n}+\vec{\Delta}_\mu$ while keeping them parallel, and then reexpressing them in terms of the basis vectors $\vec{e}_i(\vec{n}+\Delta_\mu)$. Perhaps this is confusing, and you might ask why the basis vectors $\vec{e}_i(\vec{n})$ are not each identical to the corresponding $\vec{e}_i(\vec{n}+\Delta_\mu)$. It might help to think of an ordinary vector in the plane, expressed in polar coordinates. Consider the unit basis vector $\vec{e}_r$ at the point P. If we transport it to the point P' while keeping it parallel to what it was, we arrive at the vector labelled Ge_r, which is not the same as the unit radial vector e'_r at the point P'.



[1]To a physicist, the vector space in which the matter fields live is called the representation, but to mathematicians the representation consists of the matrices M_{ij}, or more accurately the mapping from elements of the group into matrices, $G \to M(G)$.

Covariant Derivative

When a group element G acts on a vector $\vec{V} = \sum_i V_i \hat{e}_i$ which transforms under a representation M, the components of the new vector are multiplied by the matrix:

$$G : \vec{V} \to \vec{V}\,' = \sum_{ij} M_{ij}(G) V_j \hat{e}_i, \quad \text{so } V'_i = \sum_j M_{ij}(G) V_j.$$

So if G parallel transports $\vec{\phi}(\vec{n})$ from $\vec{n}$ to $\vec{n}+\vec{\Delta}_\mu$, and if we subtract this from $\vec{\phi}(\vec{n}+\Delta_\mu)$ to get the change in ϕ, we have

$$\Delta\phi = \sum_i \left[\phi_i(\vec{n}+\Delta_\mu) - \sum_j M_{ij}(G)\phi_j(\vec{n})\right]\hat{e}_i.$$

If the fields are slowly varying over the distance of one lattice spacing, which is necessary if we are to consider the lattice an approximation to the continuum, we can approximate

$$\phi_i(\vec{n}+\Delta_\mu) \approx \phi_i(\vec{n}) + a\partial_\mu \phi_i.$$

We can also assume that the group transformation that parallel transports by one lattice spacing is close to the identity, and that the Lie algebra element which generates it should be proportional to the lattice spacing a. Thus we may write $G = e^{iag\mathcal{A}}$, $M(G) = M(e^{iag\mathcal{A}}) \approx 1 + iagM(\mathcal{A})$, where $\mathcal{A}$ is an element in the Lie algebra $\mathfrak{g}$ of the gauge group $\mathcal{G}$. [We have added a parameter g which will turn out to be the fundamental charge, in order to get conventionally defined $\mathcal{A}$ fields, although sometimes that is not done, and the scale for measuring $\mathcal{A}$ is the natural one for the group.] Then we find, to first order in the lattice spacing a,

$$\Delta\phi_i = a\left(\partial_\mu\phi_i - ig\sum_j M_{ij}(\mathcal{A})\phi_j\right).$$

In the continuum limit, we define $1/a$ times this to be the **covariant derivative**, but first I must say a few words about the gauge field $\mathcal{A}$. First, as there is a different value on each link, and in the continuum limit there are four links radiating from each point, we need to be defining four fields $\mathcal{A}_\mu(\mathbf{x})$. Also, each $\mathcal{A}_\mu$ is not a single field, in general, but an element of the Lie algebra, which is a vector space. The Lie algebra for the rotation group, for example, is parameterized by a vector with three components, $\vec{\omega}$. Rotations themselves are not a gauge group, but one possible gauge group to consider is the $SU(2)$ of the electro-weak theory, which is isomorphic[2] to the rotation group. One usually uses L_i to represent a basis vector of the Lie algebra vector space, so the gauge field can be expanded as

$$\mathcal{A}_\mu(\mathbf{x}) = \sum_b A_\mu^{(b)}(\mathbf{x}) L_b.$$

This brings us to the definition of the **covariant derivative**:

$$(D_\mu\phi)_j = \partial_\mu\phi_j - ig\sum_{kb} A_\mu^{(b)} M_{jk}(L_b)\phi_k\,; \qquad D_\mu\phi = \partial_\mu\phi - igA_\mu^{(b)}M(L_b)\phi,$$

where on the right we have written the expression with implied summations and matrix and vector indices and multiplication.

Gauge Transformations

What does this have to do with local symmetry? We saw that the transformation (1), where we let G vary with x, is a symmetry for the lattice terms involving only a single site, but not for the kinetic term, $(\partial\phi)^2$, which involves cross terms such as $\sum_i \phi_i(\vec{n}+\Delta_\mu)\phi_i(\vec{n})$. These couple neighboring points, and are not invariant. But with our improved definition of $(\Delta\phi)$, the cross terms now have the form

$$\phi(\vec{n}+\Delta_\mu)\cdot M(G_L)\cdot\phi(\vec{n}),$$

where G_L is the group transformation associated with the link $(\vec{n}, \vec{n}+\Delta_\mu)$.

We can now ask what happens under the transformation in a different way. If we think of the gauge transformation $G(\mathbf{x})$ in the passive language as a change in the basis elements for the matter fields, we realize that they will also effect the rule for doing parallel transport. If G_L was the group transformation on the basis which did a parallel transport from site p to site q, with link L going from p to q, then after a change of basis by G_p at p and one by G_q at q, the way to parallel transport the new basis at p must

[2] Not exactly: the Lie algebra of $SU(2)$ is the same as the Lie algebra of the three dimensional rotation group $SO(3)$, but the actual groups differ, as is discussed when considering how spinors transform under rotations of 2π.

be $G'_L = G_q G_L G_p^{-1}$. So we now define the gauge transformation Λ, which is specified by a group element at each lattice site

$$\Lambda : \begin{cases} \phi(x_p) \to M(G_p)\cdot\phi(x_p) \\ \phi(x_q) \to M(G_q)\cdot\phi(x_q) \\ \quad G_L \to G_q G_L G_p^{-1} \end{cases}$$

This gauge transformation is a **local symmetry** of the gauge field theory. Let's verify that this is an invariance of the nearest neighbor term:

$$\begin{aligned} \phi(x_q)\cdot M(G_L)\cdot\phi(x_p) &= \phi_i(x_q)M_{ij}(G_L)\phi_j(x_p) \\ &\to M_{ik}(G_q)\phi_k(x_q)M_{ij}(G_qG_LG_p^{-1})M_{j\ell}(G_p)\phi_\ell(x_p) \\ &= \phi_k(x_q)M^{-1}_{ki}(G_q)M_{ij}(G_qG_LG_p^{-1})M_{j\ell}(G_p)\phi_\ell(x_p) \\ &= \phi_k(x_q)M_{k\ell}(G_L)\phi_\ell(x_p) = \phi(x_q)\cdot M(G_L)\cdot\phi(x_p), \end{aligned}$$

where we have used the orthogonality of $M(G_q)$ and the fact that the M's are a representation, and therefore $M^{-1}_{ki}(G_q)M_{ij}(G_qG_LG_p^{-1})M_{j\ell}(G_p) = M_{k\ell}(G_L)$.

In a continuum field theory, we consider only local gauge transformations where the group element varies differentially in the continuum limit. We may think of Λ as given by a Lie-algebra valued scalar field $\lambda(\mathbf{x}) = \sum_b \lambda^{(b)}(\mathbf{x})L_b$. Then the matter fields transform as

$$\phi(\mathbf{x}) \to \phi'(\mathbf{x}) = e^{i\sum_b \lambda^{(b)}(\mathbf{x})M(L_b)}\phi(\mathbf{x}),$$

while the gauge field itself transforms by

$$A_\mu^{(b)}(\mathbf{x}) \to A_\mu'^{(b)}(\mathbf{x}),$$

with

$$e^{iagA_\mu'^{(b)}(\mathbf{x})} = e^{i\lambda(\mathbf{x}+\frac{1}{2}a\Delta_\mu)}e^{iagA_\mu^{(b)}(\mathbf{x})L_b}e^{-i\lambda(\mathbf{x}-\frac{1}{2}a\Delta_\mu)}. \tag{2}$$

We have placed x at the middle of the link. We now expand to first order in the lattice spacing, remembering that $\lambda(\mathbf{x})$ and $\partial_\mu\lambda(\mathbf{x})$ may not commute. So we will expand the exponential rather than λ. Approximating

$$\begin{aligned} e^{iag\mathcal{A}_\mu} &\to 1 + iag\mathcal{A}_\mu, \\ e^{i\lambda(\mathbf{x}\pm\frac{1}{2}a\Delta\mu)} &\to e^{i\lambda(\mathbf{x})} \pm \frac{1}{2}a\partial_\mu\left[e^{i\lambda(\mathbf{x})}\right], \end{aligned}$$

and the same for $e^{-i\lambda(\mathbf{x})}$, and plugging these into (2), we get

$$\begin{aligned} 1 + iag\mathcal{A}'_\mu &= \left(e^{i\lambda} + \frac{1}{2}a\partial_\mu e^{i\lambda}\right)(1 + iag\mathcal{A}_\mu)\left(e^{-i\lambda} - \frac{1}{2}a\partial_\mu e^{-i\lambda}\right) \\ &= 1 + iage^{i\lambda}\mathcal{A}_\mu e^{-i\lambda} + \frac{1}{2}a\left(\partial_\mu e^{i\lambda}\right)e^{-i\lambda} - \frac{1}{2}ae^{i\lambda}\left(\partial_\mu e^{-i\lambda}\right) \end{aligned}$$

Note from $\partial_\mu\left(e^{i\lambda}e^{-i\lambda}\right) = 0$ that the third and fourth terms are equal, so we can drop the third and double the fourth, to get

$$\begin{aligned} \mathcal{A}'_\mu &= e^{i\lambda}\mathcal{A}_\mu e^{-i\lambda} + \frac{i}{g}e^{i\lambda}\partial_\mu e^{-i\lambda} \\ &= e^{i\lambda}\left(\mathcal{A}_\mu + \frac{i}{g}\partial_\mu\right)e^{-i\lambda} \end{aligned}$$

Let us now ask how this is related to the gauge transformations we know from Maxwell's theory, which look less complicated. Electromagnetism is a gauge field, but one with a very simple gauge group, that of rotations about a single fixed axis[3]. The group consists of $\mathcal{G} = \{e^{i\theta L_1}\}$ and the Lie algebra has only one generator, L_1, and is therefore isomorphic to the real line $\mathbb{R}$, and the single structure constant ${c_{11}}^1$ is zero (a counterexample to assuming that the Killing form can always be set to $2 \times \mathbb{1}$). The rotations act on charged fields, which are usually represented by complex fields Φ but in our treatment here are represented by a doublet of real fields, $(\phi_1, \phi_2) = (\text{ Re } \Phi, \text{ Im } \Phi)$. The transformation

$$\phi \to \phi' = \begin{pmatrix} \text{Re } \Phi' \\ \text{Im } \Phi' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \text{Re } \Phi \\ \text{Im } \Phi \end{pmatrix}$$

gives $\Phi' = e^{i\theta}\Phi$, so the gauge transformations are **local** changes in phase of the charged fields. The gauge transformations of fields themselves is vastly simplified by the fact that all the terms commute, so

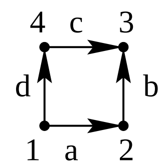
$$A'_\mu = e^{i\lambda}\left(A_\mu + \frac{i}{g}\partial_\mu\right)e^{-i\lambda} = A_\mu + g^{-1}\partial_\mu\lambda.$$

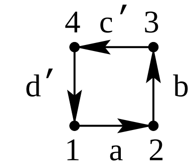
But this simplicity only holds for an **Abelian** group, one where all the generators commute, which is not enough when we wish to consider the gauge theories of the electroweak and strong interactions.

[3]These are not rotations in real space, but in some abstract space of field configurations.

Pure Gauge Terms in $\mathcal{L}$

We now know how the kinetic terms for charged fields are modified by the presence of an external gauge field, but we have not yet discussed the terms which propagate the gauge fields themselves. We need these terms in the Lagrangian to be invariant under gauge transformations. In particular this means that they cannot depend only on a single link, because we can always make a gauge transformation $G_p = G_L$ which resets the group element for a single link to 1, so there would be no dependence on the field. In fact, the simplest way to get rid of the gauge dependance of $G_a = e^{iag\mathcal{A}_x(\mathbf{x}_a)}$ on $G_2 = e^{i\lambda(\mathbf{x}_2)}$ is to premultiply it by G_b,





$$G_bG_a \to G_3G_bG_2^{-1}G_2G_aG_1^{-1} = G_3G_bG_aG_1^{-1}.$$

There is still a gauge dependence on the endpoints of the path, however, so the best thing to do is close the path. To do so, we are traversing some links backwards from the way they were defined, but from that definition in terms of parallel transport it is clear that the group element associated with taking a link backwards is the inverse of the element taken going forwards. So the group element associated with the closed path on the right (which is called a plaquette) is $G_P = G_d^{-1}G_c^{-1}G_bG_a$, which transforms under gauge transformations as

$$\begin{aligned} G_P \to G'_P &= \left(G_4G_dG_1^{-1}\right)^{-1}\left(G_3G_cG_4^{-1}\right)^{-1}G_3G_bG_2^{-1}G_2G_aG_1^{-1} \\ &= G_1G_d^{-1}G_4^{-1}G_4G_c^{-1}G_3^{-1}G_3G_bG_2^{-1}G_2G_aG_1^{-1} = G_1G_d^{-1}G_c^{-1}G_bG_aG_1^{-1} \\ &= G_1G_PG_1^{-1}. \end{aligned}$$

So the plaquette group element is not invariant but it does have a simpler and more restricted variation. In the continuum limit we expect each link's group element to be near the identity and also to have G_c differ from G_a by something proportional to the lattice spacing, so G_P should be close to the identity, the difference considered a generator in the Lie algebra. The Killing form acting on that generator will provide us with an invariant. Let us define $\mathcal{F}_{\mu\nu} = -ia^{-2}g^{-1}(G_P - 1)$ to be the field-strength tensor, where μ and ν are the directions of links a and b respectively. Let us take $\mathbf{x}$ in the center of the placquette. Expanding each link

$$G_a \approx 1 + iag\mathcal{A}_\mu(\mathbf{x} - \frac{1}{2}a\Delta_\nu) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x} - \frac{1}{2}a\Delta_\nu)$$

$$\begin{aligned}
&\approx\; 1 + iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x}) \\
G_c^{-1} &\approx\; 1 - iag\mathcal{A}_\mu(\mathbf{x}+\frac{1}{2}a\Delta_\nu) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x}+\frac{1}{2}a\Delta_\nu) \\
&\approx\; 1 - iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x}),
\end{aligned}$$

we have, to second order in a,

$$\begin{aligned}
G_P &= \left(1 - iag\mathcal{A}_\nu(\mathbf{x}) + \frac{1}{2}ia^2g\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\nu^2(\mathbf{x})\right) \\
&\quad \left(1 - iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x})\right) \\
&\quad \left(1 + iag\mathcal{A}_\nu(\mathbf{x}) + \frac{1}{2}ia^2g\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\nu^2(\mathbf{x})\right) \\
&\quad \left(1 + iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\mu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x})\right) \\
&= a^2g\left\{g\left[\mathcal{A}_\mu(\mathbf{x}),\mathcal{A}_\nu(\mathbf{x})\right] + i\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - i\partial_\nu\mathcal{A}_\mu(\mathbf{x})\right\}
\end{aligned}$$

Thus

$$\mathcal{F}_{\mu\nu}(\mathbf{x}) = \partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \partial_\nu\mathcal{A}_\mu(\mathbf{x}) - ig\left[\mathcal{A}_\mu(\mathbf{x}),\mathcal{A}_\nu(\mathbf{x})\right].$$

Note that $\mathcal{F}_{\mu\nu}$ is

- a Lie-algebra valued field, $\mathcal{F}_{\mu\nu}(\mathbf{x}) = \sum_b F_{\mu\nu}^{(b)}(\mathbf{x})L_b$.
- An antisymmetric tensor, $\mathcal{F}_{\mu\nu}(\mathbf{x}) = -\mathcal{F}_{\nu\mu}(\mathbf{x})$.
- Because the Lie algebra is defined in terms of the structure constants, ${c_{ab}}^d$ by
  $$[L_a, L_b] = i{c_{ab}}^d L_d,$$
  the field-strength tensor may also be written
  $$F_{\mu\nu}^{(d)} = \partial_\mu A_\nu^{(d)} - \partial_\nu A_\mu^{(d)} + g{c_{ab}}^d A_\mu^{(a)}A_\nu^{(b)}.$$

Before we turn to the Lagrangian, let me point out a crucial relationship between the covariant derivatives and the field-strength. If we take the commutator of covariant derivatives

$$D_\mu = \partial_\mu - igA_\mu^{(b)}L_b$$

at the same point but in different directions,

$$\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu - ig\mathcal{A}_\mu, \partial_\nu - ig\mathcal{A}_\nu] = -ig\partial_\mu\mathcal{A}_\nu - g^2\mathcal{A}_\mu\mathcal{A}_\nu - (\mu\leftrightarrow\nu) \\
&= -g^2\left[\mathcal{A}_\mu,\mathcal{A}_\nu\right] - ig\partial_\mu\mathcal{A}_\nu + ig\partial_\nu\mathcal{A}_\mu \\
&= -ig\mathcal{F}_{\mu,\nu}.
\end{aligned}$$

Notice that although the covariant derivative is in part a differential operator, the commutator has neither first or second derivatives left over to act on whatever appears to the right. It does need to be interpreted, however, as specifying a representation matrix that will act on whatever is to the right.

Now consider adding to the Lagrangian a term proportional to $\beta(\mathcal{F}_{\mu\nu},\mathcal{F}^{\mu\nu}) = 2\sum_b F_{\mu\nu}^{(b)}F^{(b)\,\mu\nu}$. I have assumed the generators L_i have been normalized so that the Killing form $\beta(L_i,L_j) = 2\delta_{ij}$, and the stucture constants are totally antisymmetric[4]. We know that under a gauge transformation $\mathcal{F}_{\mu\nu} \to e^{i\lambda}\mathcal{F}_{\mu\nu}e^{-i\lambda}$. If λ is infinitesimal, $\mathcal{F}_{\mu\nu} \to \mathcal{F}_{\mu\nu} + i[\lambda,\mathcal{F}_{\mu\nu}] = \left\{F_{\mu\nu}^{(d)} - \lambda^{(a)}F_{\mu\nu}^{(b)}{c_{ab}}^d\right\}L_d$, so

$$\delta\beta(\mathcal{F}_{\mu\nu},\mathcal{F}^{\mu\nu}) \;=\; -2\lambda^{(a)}F_{\mu\nu}^{(b)}{c_{ab}}^dF^{(d)\,\mu\nu} = 0$$

where the expression vanishes because ${c_{ab}}^d$ is antisymmetric under interchange of b and d but $F_{\mu\nu}^{(b)}F^{(d)\,\mu\nu}$ is symmetric under the same interchange (and we are summing on b and d). As $\beta(\mathcal{F}_{\mu\nu},\mathcal{F}^{\mu\nu})$ doesn't change to first order under infinitesimal transformations, it also doesn't change under the finite transformations they generate.

Equations of Motion for the Gauge Fields

We choose the normalization of the A fields so that the pure gauge term in the Lagrangian density is $-\frac{1}{4}F_{\mu\nu}^{(b)}F^{(b)\,\mu\nu}$. Suppose we also have Dirac matter fields transforming under a representation $t_{ij}^b = M_{ij}(L_b)$ of the group, and perhaps some scalar fields as well, transforming under a (possibly) different representation $\bar{t}_{ij}^b = \bar{M}_{ij}(L_b)$, where the bars here only represent a different representation, not any kind of conjugation. The gauge fields come into the matter terms in the Lagrangian because, in order to maintain local gauge invariance, all derivatives need to be replaced by covariant derivatives. Thus the potential terms for matter fields in $\mathcal{L}$ will not be involved in the equations

[4]See `groups.pdf` and `adjnote.pdf` in `http://www.physics.rutgers.edu/~shapiro/616/`

of motion of the gauge fields, and we need only look at

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{(b)}_{\mu\nu}F^{(b)\,\mu\nu} + i\bar{\psi}\gamma^\mu\left(\partial_\mu - igA^{(b)}_\mu t^b\right)\psi \\ &+\frac{1}{2}\left[\left(\partial_\mu - igA^{(b)}_\mu \bar{t}^b\right)\phi\right]^T\left[\left(\partial^\mu - igA^{(b)\,\mu}\bar{t}^b\right)\phi\right].\end{aligned}$$

Let us analyze the classical mechanics of the gauge fields. First we find the canonical momentum conjugate to $A^{(b)}_\mu(\mathbf{x})$, which is

$$\pi^{(b)\,\mu}(\mathbf{x}) = \frac{\delta\mathcal{L}}{\delta\dot{A}^{(b)}_\mu}(\mathbf{x}) = -\frac{1}{2}F^{(b)\,\mu\nu}\frac{\delta F^{(b)}_{\mu\nu}}{\delta\dot{A}^{(b)}_\mu} = -F^{(b)\,0\mu}(\mathbf{x}).$$

Notice that

- The matter field terms do not contribute to the canonical momenta, because they depend on $\mathcal{A}_\mu$ but not its time derivative.
- The momentum conjugate to $\mathcal{A}_0$ is **identically** zero.

Momenta are not supposed to be identically zero in ordinary classical mechanics, they are supposed to be substitutes for velocities, *i.e.* time derivatives of the coordinates. The N coordinates and N momenta of an N degrees-of-freedom problem are supposed to span a $2N$ dimensional phase space. $\pi^0 = 0$ is not an equation of motion, it is a constraint. To properly handle such a situation we would need either to eliminate the constrained degrees of freedom or use some fancy techniques not usually discussed in classical mechanics courses. The canonical form of quantum mechanics would seem to require zero to not commute with $\mathcal{A}_0$, which is too strange to contemplate even in quantum mechanics. But it turns out that this situation can be handled somewhat straightforwardly in the path integral formulation of quantum mechanics, as we shall see.

In the classical mechanics of a field theory we generally define a generalization of the momentum,

$$\Pi^\nu_i(\mathbf{x}) = \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_i}, \qquad \Pi^0_i(\mathbf{x}) = \pi_i(\mathbf{x})$$

for each field ϕ_i. The $\nu = 0$ component is the conjugate momentum, but all components enter the Euler-Lagrange equations

$$\partial_\nu\Pi^\nu = \frac{\delta\mathcal{L}}{\delta\phi}.$$

Here our fields are not scalars but have additional indices b and μ because $\mathcal{A}$ is both Lie-algebra-valued and a vector. So the Euler-Lagrange equation is

$$\partial_\nu\Pi^{(b)\,\mu;\nu} - \frac{\delta\mathcal{L}}{\delta A^{(b)}_\mu} = 0,$$

where

$$\Pi^{(b)\,\mu;\nu} = \frac{\delta\mathcal{L}}{\delta\left(\partial_\nu A^{(b)}_\mu\right)} = -F^{(b)\,\nu\mu}.$$

Let's turn to evaluating the right-hand side of the Euler-Lagrange equation, $\delta\mathcal{L}/\delta A^{(b)}_\mu$. Recall that the derivative intended here is to look for terms depending directly on $A^{(b)}_\mu$ considering the derivatives fixed. We will need to differentiate the field-strength

$$F^{(c)}_{\mu\nu} = \partial_\mu A^{(c)}_\nu - \partial_\nu A^{(c)}_\mu + gc_{ab}{}^c A^{(a)}_\mu A^{(b)}_\nu,$$

but only the last term depends on undifferentiated gauge fields, so

$$\frac{\delta F^{(c)}_{\mu\nu}}{\delta A^{(b)}_\rho} = gc_{ab}{}^c\left(A^{(a)}_\mu\delta^\rho_\nu - A^{(a)}_\nu\delta^\rho_\mu\right),$$

and thus

$$\frac{\delta}{\delta A^{(b)}_\rho}\left[-\frac{1}{4}F^{(c)}_{\mu\nu}F^{(c)\,\mu\nu}\right] = -gc_{ab}{}^c F^{(c)\,\mu\rho}A^{(a)}_\mu.$$

The contributions to $\delta\mathcal{L}/\delta A^{(b)}_\rho$ from the matter terms are more straightforward — each D_ρ will contribute a $-igt^b$ or equivalent, so

$$\frac{\delta}{\delta A^{(b)}_\rho}\left\{+i\bar{\psi}\gamma^\mu D_\mu\psi + \frac{1}{2}\left[D_\mu\phi\right]^T D_\mu\phi\right]\right\} = g\bar{\psi}\gamma^\rho t^b\psi + ig\phi^T\bar{t}^b D^\rho\phi,$$

where I have used the fact that the representations of the generators $\bar{t}$ for real unitary representations are antisymmetric imaginary matrices.

Define the current

$$j^a_\nu = \bar{\psi}\gamma_\nu t^a\psi + i\phi^T\bar{t}^a D_\nu\phi,$$

so that we have found

$$\partial_\nu F^{(b)\,\nu\mu} - gc_{ab}{}^c F^{(c)\,\nu\mu}A^{(a)}_\nu + gj^{b\,\mu} = 0. \tag{3}$$

The appearence of the first two terms is reminiscent of a covariant derivative, but we have not explicitly defined the covariant derivative acting on Lie-algebra valued fields, because we have not explicitly explored their representation under the group. While the local gauge transformation of the gauge field is more complicated than can be described by a representation of the gauge group, under a *global* gauge transformation it or any other Lie-algebra valued field transforms as

$$e^{i\lambda^c L_c} : \mathcal{A} \to e^{i\lambda^c L_c} \mathcal{A} e^{-i\lambda^c L_c} = M^{\rm adj}_{ab}\left(e^{i\lambda^c L_c}\right) A^b L_a,$$

which is the adjoint represenation. Differentiating wrt λ_c and setting λ to zero gives

$$[L_c, A^b L_b] = M^{\rm adj}_{ab}\left(L_c\right) A^b L_a = i{c_{cb}}^a A^b L_a,$$

so the appropriate representation for the covariant derivative of a Lie-algebra valued field is

$$M^{\rm adj}_{ab}\left(L_c\right) = i{c_{cb}}^a.$$

$M^{\rm adj}_{ab}\left(L_c\right)$ is called the **adjoint** representation, and is always of the same dimension as the Lie algebra itself.

As discussed above[5] we have normalized our generators L_b so that the structure constants are totally antisymmetric, and we can substitute

$$iM^{\rm adj}_{ab}\left(L_c\right) = -{c_{cb}}^a = {c_{ab}}^c$$

in the expression (3), giving

$$\partial_\rho F^{(a)\,\rho\mu} - igA^{(c)}_\rho M^{\rm adj}_{ab}(L_c) F^{(b)\,\rho\mu} + gj^{a\,\mu} = D_\rho F^{(a)\,\rho\mu} + gj^{a\,\mu} = 0.$$

Now that we have defined the covariant derivative of a Lie algebra valued field,

$$D_\mu \lambda^{(b)} = \partial_\mu \lambda^{(b)} - igA^{(c)}_\mu M^{\rm adj}_{bd}(L_c)\lambda^{(d)},$$

we may note that for an infinitesimal gauge transformation,

$$A^{(b)}_\mu \to A'^{(b)}_\mu = A^{(b)}_\mu + i\lambda^d A^{(c)}_\mu (i{c_{dc}}^b) + \frac{1}{g}\partial_\mu \lambda^{(b)} = A^{(b)}_\mu + \frac{1}{g}\left(D_\mu \lambda\right)^{(b)}.$$

The theory we have just defined, the gauge theory based on a non-Abelian Lie group, is known as Yang-Mills theory.

[5] And more extensively in my notes on "Adjoint Representation, Killing forms and the antisymmetry of ${c_{ij}}^k$", `http://www.physics.rutgers.edu/grad/616/adjnote.pdf`.

Inadequacy of the Equations of Motion

Field configurations which are gauge transformations of each other cannot be distinguished. As local symmetries can affect the asymptotic states in a scattering experiment, even to the extent of affecting the fields representing just one of the particles, it is clear that field configurations related by a gauge transformation *do not describe different physical states!* Thus there appear to be field excitations describing particle states which are, in fact, unphysical. These degrees of freedom of the fields are also undetermined by the physics. For example, if we take the $g \to 0$ limit of the gauge theory,

$$\begin{aligned} F^{(c)}_{\mu\nu} &= \partial_\mu A^{(c)}_\nu - \partial_\nu A^{(c)}_\mu \\ \partial^\mu F^{(c)}_{\mu\nu} &= \partial^\mu \partial_\mu A^{(c)}_\nu - \partial_\nu \partial^\mu A^{(c)}_\mu = 0. \end{aligned}$$

If we expand the A field in fourier modes as we did for ϕ,

$$A^{(c)}_\mu(\mathbf{x}) = \int \frac{d^4k}{(2\pi)^4} e^{-ik_\nu x^\nu} \tilde{A}^{(c)}_\mu(\mathbf{k}),$$

we find the equations of motion

$$k^2 \tilde{A}^{(c)}_\mu(\mathbf{k}) - k_\mu k^\nu \tilde{A}^{(c)}_\nu(\mathbf{k}) = 0 \tag{4}$$

Let us compare this equation to the corresponding equation for the scalar field,

$$(k^2 - m^2)\tilde{\phi}(\mathbf{k}) = 0.$$

The scalar field equation tells us that, in the absence of a source, $\tilde{\phi}(\mathbf{k}) = 0$ unless $k^2 = m^2$. That is, only on-shell particles can propagate in free space, as we would expect. But for each four-vector $\mathbf{k}$ the equations for the four components of the gauge field $\tilde{A}^{(c)}_\nu(\mathbf{k})$ only provide three real constraints, for if we project out one component of Eq. (4) by multiplying by k^μ, we get the identity $(k^2 - k^2)k^\nu \tilde{A}^{(c)}_\nu(\mathbf{k}) = 0$, which does not provide a constraint (a partial differential equation) for the gauge field. Thus the equations of motion are not adequate to determine the evolution of the gauge field.

This should come as no surprise. Ordinary mechanics is deterministic, in the sense that if you know enough about the state at time zero (including time derivatives) you can predict the degrees of freedom in the future. But with local gauge invariance we can make a gauge transformation which has no effect for times before $t = 1$, but can change the values of the fields in the future. This doesn't mean the physics is non-deterministic, only that there is spurious information in the fields which is neither determined nor physical.

If we believe the equations should determine the future physical state even if not a unique representation of it, we can ask if we might add equations which would make the equations deterministic. The result of propagating the fields would then represent the correct physical state. Even if there is arbitrariness in the equations we add, so the future fields will be somewhat arbitrary, the physical state we get should be uniquely determined.

Consider again the free theory. With $g = 0$ each Lie-algebra component is independent of the others, so we might as well consider just the Maxwell situation, with only one component A_μ. The field is gauge equivalent to any other field A'_μ with[6]

$$A'_\mu = A_\mu + \partial_\mu \lambda$$

If I like, I can insist that my new field be four-divergenceless:

$$\partial^\mu A'_\mu = \partial^\mu A_\mu + \partial^2 \lambda = 0$$

by solving the equation $\partial^2 \lambda(\mathbf{x}) = \rho(\mathbf{x})$, where the source term $\rho(\mathbf{x}) = -\partial^\mu A_\mu(\mathbf{x})$. So I can add the fourth differential equation

$$\partial^\mu A_\mu = 0$$

to the equations of motion, to get a deterministic set. The additional equation is called the **Lorenz gauge condition**. Any solution to the previous set of equations will be equivalent, up to a gauge transformation, to a solution of this enlarged set.

The Lorenz gauge condition is just one possibility — more generally we might imagine taking some local function of the gauge field, and requiring it to vanish[7]: $G(\mathcal{A}_\mu) = 0$. It should have the property that for any gauge configuration $\mathcal{A}_\mu$, there should be a unique gauge transformation, say with generator $\lambda(\mathbf{x})$, such that under this transformation $\Lambda : \mathcal{A}_\mu \mapsto \mathcal{A}_\mu^\lambda$ with $G(\mathcal{A}_\mu^\lambda) = 0$. Then adding the equation $G(\mathcal{A}_\mu) = 0$ would provide the deterministic equation to keep the evolution of the gauge field determined, and still represent the right physical state.

That statement helps us do classical mechanics, but for quantum mechanics we need to sum over all physical states independent of the equations

[6]I have rescaled the required λ by a factor of g from our previous expression, as all this is intended to describe the $g = 0$ approximation.

[7]Here G is some function of $\mathcal{A}_\mu$ and perhaps its first derivatives, not a group element.

of motion. We want the functional integral to count each physically distinct field configuration once, or at least equally. We could insure that each such state appears once in the integration volume, by inserting a Dirac delta function imposing the $G = 0$ condition, $\delta(G(\mathcal{A}_\mu))$. (This needs to be interpreted as a delta function *at each point of spacetime*.) But while the delta function insures each physical field occurs only as one gauge field configuration, it doesn't ensure different physical configurations are counted equally!

Instead, we will use a trick due to Faddeev and Popov. Basically it breaks up the integration regions (in the very large space of all gauge field configurations) into equivalence classes under the gauge group, and then factors out the integration within each class. This involves the idea of integration over a group.

Integration over a Group

For a continuous group, a group element g can be specified by a set of coordinates in some fashion. This might be the coefficients of the generators in the exponential representing that element, or it might be something like the Euler angles which determine a rotation. Whatever the coordinates ν_i are, integration over the group requires a measure, $d\mu(\nu) = h(\nu) d\nu_1 \dots d\nu_n$ for an n-dimensional Lie group G. The measure is invariant under the group if, for any function f on G,

$$\int_G d\mu(\nu) f(g(\nu)) = \int_G d\mu(\nu) f(g' g(\nu)) = \int_G d\mu(\nu) f(g(\nu) g')$$

for every fixed element g' of the group. This measure is uniquely determined by the measure near the identity, because any infinitesimal neighborhood of any point $g \in G$ can be mapped into an infinitesimal neighborhood of the identity by multiplying by g^{-1}. So the invariant measure is determined up to a constant for any group, and is called the **Haar** or **Hurwitz** measure on the group. For a semisimple Lie group with the generators normalized as we have discussed, the measure near the identity is just $\prod d\lambda^{(b)}$.

Inserting a Dirac delta function

We also should review the use of Dirac delta functions. If X is an n-dimensional space with coordinates x_i, then an n-dimensional delta function has the property

$$\int_X d^n x_i \delta(x_i - c_i) f(\mathbf{x}) = f(\mathbf{c}),$$

where $\mathbf{c}$ is an arbitrary point in X and f an arbitrary function on X. Somewhat more generally, if h is a 1-1 onto function mapping the space X into the space Y, ($y_i = h_i(\mathbf{x})$) and $\mathbf{c}$ is now an arbitrary point in Y, then the equivalent is

$$\int_X d^n x_i \delta(h_i(\mathbf{x}) - c_i) \det\left(\frac{\partial h_i}{\partial x_j}\right) f(\mathbf{x}) = f(h^{-1}\mathbf{c}).$$

This can be shown by changing variables from x to y, which requires the Jacobian. We don't actually need h to be 1-1 onto on the whole space — all that is really needed is that only the one point, $h^{-1}(\mathbf{c})$, is mapped to $\mathbf{c}$, and h is 1-1 and differentiable in some neighborhood of that point, so that the Jacobian is well defined.

Now consider the space X to be the space of local gauge transformations Λ, and h to be the map that applies our gauge condition to the gauge transform of a fixed gauge field $\mathcal{A}$, so $h(\lambda) = G(\mathcal{A}^\lambda)$. We have already assumed our function G is such that only one gauge transformation λ gives an $\mathcal{A}^\lambda$ upon which G vanishes, so it satisies our condition, and

$$\int \mathcal{D}\lambda \delta(G(\mathcal{A}^\lambda)) \det\left(\frac{\delta G(\mathcal{A}^\lambda)}{\delta\lambda}\right) f(\mathcal{A}^\lambda) = f(\mathcal{A}^\alpha),$$

where α is the single gauge transformation for which $G(\mathcal{A}^\alpha) = 0$.

We will naturally assume that our definition of $\int \mathcal{D}\lambda$ is the invariant measure. Then we claim that the last expression is independent of gauge transformations on the fixed gauge field $\mathcal{A}$. If g_λ represents the gauge transformation $e^{i\lambda(\mathbf{x})}$, and if $g_\beta : \mathcal{A} \mapsto \mathcal{A}' = \mathcal{A}^\beta$ for some gauge transform $e^{i\beta}$, then $\mathcal{A}'^\lambda = g_\lambda g_\beta \mathcal{A}$, so, with $\mathcal{A}$ considered fixed,

$$\int \mathcal{D}\lambda \delta(G(\mathcal{A}'^\lambda)) \det\left(\frac{\delta G(\mathcal{A}'^\lambda)}{\delta\lambda}\right) f(\mathcal{A}'^\lambda)$$

is of the form $\int_G d\mu(\nu) f(g(\nu)g')$ we discussed above, provided that the integration over gauge group elements is the group invariant Hurwitz measure, which we have assumed. Here g' is the gauge transformation by β, and g that by λ, so $\mathcal{A}'^\lambda = gg'(\mathcal{A})$. But the invariance of the measure allows us to drop the g', so

$$\int \mathcal{D}\lambda \delta(G(\mathcal{A}'^\lambda)) \det\left(\frac{\delta G(\mathcal{A}'^\lambda)}{\delta\lambda}\right) f(\mathcal{A}'^\lambda) = f(\mathcal{A}^\alpha),$$

where again α is the single gauge transformation for which $G(\mathcal{A}^\alpha) = 0$.

Let us define the functional

$$\Delta[\mathcal{A}] = \int \mathcal{D}\lambda \delta(G(\mathcal{A}^\lambda)),$$

which corresponds to

$$f(\mathcal{A}^\lambda) = \det\left(\frac{\delta G(\mathcal{A}^\lambda)}{\delta\lambda}\right)^{-1}.$$

Thus we might add a factor of $(\Delta[\mathcal{A}])^{-1} \int \mathcal{D}\lambda \delta(G(\mathcal{A}^\lambda))$ inside the functional integral over $\mathcal{D}\mathcal{A}$. Thus we may write the functional integral we want as

$$\begin{aligned}\int \mathcal{D}\mathcal{A}\, e^{iS[\mathcal{A}]} &= \int \mathcal{D}\mathcal{A}\, (\Delta[\mathcal{A}])^{-1} e^{iS[\mathcal{A}]} \int \mathcal{D}\lambda \delta(G(\mathcal{A}^\lambda)) \\ &= \int \mathcal{D}\lambda \int \mathcal{D}\mathcal{A}\, (\Delta[\mathcal{A}^\lambda])^{-1} e^{iS[\mathcal{A}^\lambda]} \delta(G(\mathcal{A}^\lambda))\end{aligned}$$

where we have used the fact that both Δ and S are gauge invariant functionals of $\mathcal{A}$.

Now for fixed λ consider changing integration variables from $\mathcal{A}$ to $\mathcal{A}^\lambda$. The Jacobian for such a change is easily evaluated from

$$(\mathcal{A}^\lambda)^{(b)}_\mu L_b = A^{(a)}_\mu e^{i\lambda} L_a e^{-i\lambda} + \frac{i}{g} e^{i\lambda} \partial_\mu e^{-i\lambda} = A^{(a)}_\mu M^{\mathrm{adj}}_{ab}(e^{i\lambda}) L_b + \frac{i}{g} e^{i\lambda} \partial_\mu e^{-i\lambda}.$$

The last term is independent of $\mathcal{A}$ and doesn't contribute to the Jacobian, and the adjoint representation is orthogonal and so has determinant 1. Thus we can simply replace $\mathcal{D}\mathcal{A}$ with $\mathcal{D}\mathcal{A}^\lambda$, and then, as $\mathcal{A}^\lambda$ is a dummy integration variable, we can replace it by $\mathcal{A}$ everywhere. Now the integral over the gauge transformation is of an integrand independent of the transformation, so it just gives the volume of the gauge group $V_\mathcal{G}$, which will be irrelevant:

$$\int \mathcal{D}\mathcal{A}\, e^{iS[\mathcal{A}]} = V_\mathcal{G} \int \mathcal{D}\mathcal{A}\, (\Delta[\mathcal{A}])^{-1} e^{iS[\mathcal{A}]} \delta(G(\mathcal{A})). \qquad (5)$$

In this form, we appear to be integrating only over gauge fields which satisfy the gauge condition $G(\mathcal{A}) = 0$. That is in line with our original idea that we should only integrate over physically distinct gauge fields, though we see that we need a correction to the naïve measure given by $(\Delta[\mathcal{A}])^{-1}$, which is known as the Faddeev-Popov determinant[8]. This form is not convenient, however, because it is easier to integrate over all $\mathcal{A}$ fields than over those that satisfy a gauge condition. We can convert Eq. (5) to such a form by noticing that the left hand side is independent of the gauge condition. So we could choose an arbitrary condition at each point in space-time,

$$G(A^{(b)}) = \partial^\mu A^{(b)}_\mu - \omega^{(b)}(\mathbf{x}),$$

and we can multiply the functional integral by

$$\int \mathcal{D}\omega^{(b)} \exp\left(\frac{-i}{2\xi}\int d^4x\, \omega^2(\mathbf{x})\right),$$

(with ξ some numerical constant) which evaluates to an irrelevant constant factor. Bringing this inside the functional integral over $\mathcal{A}$ and using the $\delta(G(\mathcal{A})) = \delta(\partial^\mu A^{(b)}_\mu - \omega^{(b)}))$ to do the integral over $\mathcal{D}\omega^{(b)}$, we arrive at

$$\int \mathcal{D}\mathcal{A}\, e^{iS[\mathcal{A}]} \propto \int \mathcal{D}\mathcal{A}\, (\Delta[\mathcal{A}])^{-1} \exp\, i\int d^4x \left(\mathcal{L}_{YM}[\mathcal{A}] - \frac{1}{2\xi}(\partial^\mu A^{(b)}_\mu)^2\right). \quad (6)$$

This appears to be what we would get by adding the gauge-variant term $-\frac{1}{2\xi}(\partial^\mu A^{(b)}_\mu)^2$ to the Lagrangian density, except for the Faddeev-Popov determinant.

We now need to turn to the Faddeev-Popov determinant. This is

$$\Delta^{-1}[\mathcal{A}] = \det \frac{\delta\, \partial^\mu (A^\lambda_\mu)^{(b)}}{\delta\lambda^{(a)}}.$$

We shall return to the question of evaluating this for Yang-Mills theory after we study fermions, and their associated Grassman variables. At this point we will evaluate the determinant for an Abelian theory, such as Maxwell's, for which $\delta A^\lambda_\mu = g^{-1}\partial_\mu\lambda$. Then the determinant is independent of all fields, and may be considered a constant and dropped.

The upshot of all this is that, for Abelian theory, the gauge invariant Lagrangian may be replaced by one with an additional, gauge variant term, at least in evaluating $Z[0]$. There are additional complications for $Z[J]$ unless the source term J is gauge invariant — we will return to that after we find the photon propagator and interactions.

[8] That is, Δ^{-1} is the Faddeev-Popov determinant, not Δ.

Feynman Propagator for Gauge Fields

We have seen that, for Maxwell theory, in place of the free Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}(\partial^\mu A_\nu)(\partial_\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\nu)(\partial_\nu A_\mu)$$

we now have the gauge-fixed lagrangian

$$\mathcal{L}_{\rm GF} = -\frac{1}{2}(\partial^\mu A_\nu)(\partial_\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\nu)(\partial_\nu A_\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)^2.$$

In homework 3, problem 2, you will show that the gauge field propagator with this Lagrangian is

$$\tilde{D}^{\mu\nu}_F(\mathbf{k}) = \frac{-i}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right),$$

which actually also has a δ^a_b in Lie-algebra components.

We also need to consider interactions. Recall that for the scalar field the $-\lambda\phi^4/4!$ in the Lagrangian, when treated perturbatively, gave rise to vertices in Feynman diagrams with four legs attached, representing $-i\lambda(2\pi)^4\delta^4(\sum \mathbf{p})$. For the gauge fields we have interactions with scalars and with spinors (which we have not yet considered), but we also have self-interactions among the gauge fields. The full expansion of the pure gauge term is

$$\begin{aligned}
-\frac{1}{4}F^{(a)\,\mu\nu}F^{(a)}_{\mu\nu} &= -\frac{1}{2}(\partial^\mu A^{(a)}_\nu)(\partial_\mu A^{(a)\,\nu}) + \frac{1}{2}(\partial_\mu A^{(a)\,\nu})(\partial_\nu A^{(a)}_\mu) \\
&\quad -\frac{1}{2}g c_{ab}{}^d A^{(a)\,\mu}A^{(b)\,\nu}(\partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu) \\
&\quad -\frac{1}{4}g^2 c_{ab}{}^d c_{ef}{}^d A^{(a)}_\mu A^{(b)}_\nu A^{(e)\,\mu}A^{(f)\,\nu}.
\end{aligned}$$

The terms not already considered for the free Lagrangian are thus

$$\mathcal{L}_{\rm GI} = -g c_{ab}{}^d A^{(a)}_\mu A^{(b)}_\nu \partial^\mu A^{\nu\,(d)} - \frac{1}{4}g^2 c_{ab}{}^d c_{ef}{}^d A^{(a)}\cdot A^{(e)}A^{(b)}\cdot A^{(f)}.$$

These terms in perturbation theory will generate vertices in the Feynman graphs, the first with three gauge particles entering, the second with four.

There are also terms expressing the interaction of gauge fields with matter fields,

$$\mathcal{L}_{\rm GM} = g\bar\psi\gamma^\mu t^b\psi A^{(b)}_\mu + ig\phi^T\bar{t}^b\partial_\mu\phi A^{(b)}_\mu + \frac{1}{2}g^2\phi^T\bar{t}^b\bar{t}^a\phi A^{(b)}_\mu A^{(a)\,\mu}.$$

The first of these represents a gauge particle meeting a fermion line in the Feynman graph. The second and third terms represent one or two gauge particles, respectively, being emitted (or absorbed) at a point along a scalar line.

The gauge interactions with spinors are both simpler and more interesting (in their physics applications) than for scalars, so it is now time to turn to fermions.