Here we evaluate the “area” of the surface of a ball of radius 1 in \( D \) dimensions, that is, the (hyper) volume of a \( D - 1 \) dimensional sphere \( S_{D-1} \).

To do so we also need to evaluate the Euler Gamma function
\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad \text{for } \text{Re} \ z > 0.
\]

Note that \( \Gamma(1) = 1 \) and
\[
\Gamma(z + 1) = -\int_0^\infty t^zd(e^{-t}) = t^ze^{-t}\bigg|_0^\infty + z\int_0^\infty t^{z-1}e^{-t}dt = z\Gamma(z)
\]
for \( \text{Re} \ z > 0 \). We can evaluate \( \Gamma \) for half-integer arguments and simultaneously the volume of a \( D - 1 \) sphere by evaluating this integral in \( D \) dimensional Euclidean space:
\[
I = \int_{\mathbb{R}^D} e^{-r^2}.
\]

If we do the integral using cartesian coordinates,
\[
I = \prod_{i=1}^D \left( \int_{-\infty}^{\infty} dr_i \right) e^{-\sum r_i^2} = \prod_{i=1}^D \left( \int_{-\infty}^{\infty} dr_i e^{-r_i^2} \right) = \left( \int_{-\infty}^{\infty} du e^{-u^2} \right)^D.
\]

The integral in the last expression is
\[
\int_{-\infty}^{\infty} du e^{-u^2} = 2 \int_0^{\infty} du e^{-u^2} \to \int_0^{\infty} dt \ e^{-\frac{t^2}{4}} = \Gamma\left(\frac{1}{2}\right),
\]
so
\[
I = \left( \Gamma\left(\frac{1}{2}\right) \right)^D.
\]

On the other hand \( e^{-r^2} \) is hyperspherically symmetric, so
\[
I = S_{D} \int_0^{\infty} r^{D-1} e^{-r^2} dr = \frac{1}{2} S_{D} \int_0^{\infty} t^{\frac{D}{2}-1} e^{-t} dt = \frac{1}{2} S_{D} \Gamma\left(\frac{D}{2}\right),
\]
where \( S_{D} = \int d\Omega_{D} \) is the surface area of a unit ball in \( D \) dimensions. Thus we have
\[
S_{D} = \frac{1}{2} S_{D} \Gamma\left(\frac{D}{2}\right) = \left( \Gamma\left(\frac{1}{2}\right) \right)^D.
\]

For \( D = 2 \) we know, of course, that the surface of a 2-ball, that is a circle of radius 1, has “volume” \( 2\pi \), so
\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.
\]

Thus for all \( D \) we have
\[
S_{D} = \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}.
\]

From \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \) and \( \Gamma(1) = 1 \), and from the recursion relation \( \Gamma(z + 1) = z\Gamma(z) \), we can evaluate
\[
\Gamma\left(\frac{D}{2}\right) = \sqrt{\pi}, 1, \frac{\sqrt{\pi}}{2}, 1, \frac{3\sqrt{\pi}}{4}, 2, \ldots \quad \text{for } D = 1, 2, \ldots,
\]
and thus
\[
S_{D} = 2, 2\pi, 4\pi, 2\pi^2, \frac{8\pi^2}{3}, \ldots \quad \text{for } D = 1, 2, \ldots.
\]

\[\text{Note: } -\sqrt{\pi} \text{ as the integrand is clearly positive definite.}\]