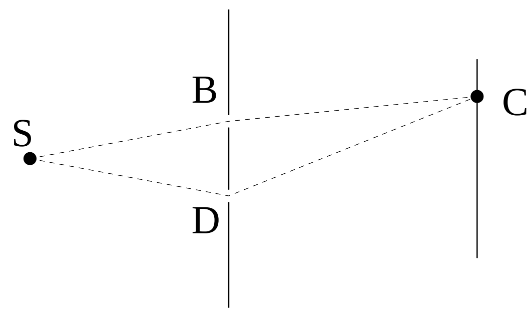
Feynman Path Integral Formulation of Quantum Mechanics

[This is based on my 1980 lecture notes for 616]

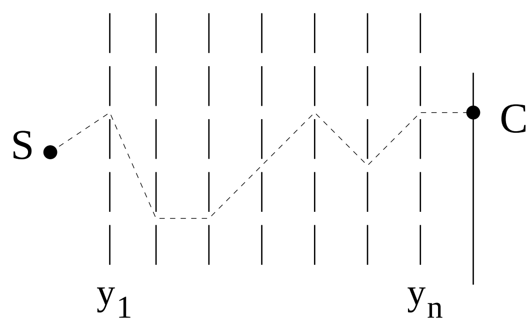
If your introduction to quantum mechanics was like mine, the thing that convinced you that classical ideas couldn't work was an experiment in which a particle is given a choice ot two holes to go through before hitting a screen. We are told that the intensity observed at C is the square of the amplitude A, which is a sum of amplitudes for $S \rightarrow B \rightarrow C$ and $S \rightarrow D \rightarrow C$. Each of these amplitudes is a product of an amplitude for $S \rightarrow$ hole and an amplitude for hole $\rightarrow C$. The astounding thing about this double slit experiment is that the amplitudes for the different holes is what gets added, rather than the probabilities. In wave terms, the amplitudes are added, complete with phases, and the intensity shows interference.



If there were more than two slits, the amplitude would be the sum over all of them. Now consider a system of many screens, $i = 1 \ldots n$, each with many slits, $j = 1 \ldots m_i$. Clearly the amplitude $S \rightarrow C$ is just

$$\mathcal{A}(C, S) = \sum_{y_1=1}^{m_1} \cdots \sum_{y_n=1}^{m_n} \prod_{i=0}^{n} \mathcal{A}(y_{i+1}, y_i),$$

where $y_0 = S$ and $y_{n+1} = C$. One term in the sum corresponds to the path shown. This is just the mathematical statement that the total amplitude is the sum of the amplitudes for each possible path.



If the screens were so completely filled with slits that nothing was left of them, the sum over slits would now be over all possible y values, and the sums would become integrals over each y,

$$\mathcal{A}(C,S) = \int dy_n \mathcal{A}(y_{n+1},y_n) \int dy_{n-1}\mathcal{A}(y_n,y_{n-1}) \cdots \int dy_1 \mathcal{A}(y_2,y_1)\mathcal{A}(y_1,y_0)$$

In this example we were able to think that the particle is always moving forward, so that the different screens at $x_1 \ldots x_n$ also represent different times, and we might also consider that $\mathcal{A}(y_{i+1},y_i)$ represents the amplitude for a particle at position y_i at time t_i to wind up at y_{i+1} at time t_{i+1}. Such an amplitude, for a particle with a time-invariant Hamiltonian, is given in the operator formulation of quantum mechanics by the unitary time-evolution operator, with coordinate-space matrix elements

$$U(y_f,y_i;t_f,t_i) = \langle y_f | e^{-iH\,[t_f-t_i]/\hbar} | y_i \rangle.$$

This operator evolves the wave-function in time

$$\psi(y_f,t_f) = \int dy\, U(y_f,y,t_f,t_i)\psi(y,t_i), \tag{1}$$

and therefore is unitary and has the property

$$U(y_f,y_i,t_f,t_i) = \int dy\, U(y_f,y,t_f,t')U(y,y_i,t',t_i)$$

which expresses the fact that evolving from $t_i \to t'$ and then from $t' \to t_f$ is evolving from $t_i \to t_f$.

In particular,

$$\begin{aligned} \frac{d}{dt}U(y,y_i;t,t_i) &= \lim_{\Delta t\to 0} \frac{U(y,y_i;t+\Delta t,t_i) - U(y,y_i;t,t_i)}{\Delta t} \\ &= \frac{1}{\Delta t}\int dy' \left(U(y,y';t+\Delta t,t) - \delta(y-y')\right) U(y',y_i;t,t_i). \end{aligned}$$

But to first order in Δt,

$$U(y,y';t+\Delta t,t) = \langle y|1 - iH\Delta t/\hbar|y'\rangle = \delta(y-y') - i(\Delta t/\hbar)\langle y|H|y'\rangle,$$

so

$$i\hbar\frac{d}{dt}U(y,y_i;t+\Delta t,t_i) = \int dy' \langle y|H|y'\rangle U(y',y_i;t,t_i),$$

which is to say, from Eq. 1,

$$i\hbar\frac{d\psi}{dt} = H\psi,$$

the Schrödinger equation.

Now we need to examine the time-evolution over infinitesimal time intervals. The Hamiltonian is, in general, a function of the generalized coordinates $\vec{q}$ and the momentum $\vec{p}$, so we wish to evaluate

$$\langle \vec{q}\,'|\, e^{-iH\Delta t/\hbar}\, |\vec{q}\rangle \approx \langle \vec{q}\,'|\, \vec{q}\rangle - \frac{i\Delta t}{\hbar} \langle \vec{q}\,'|\, H(\vec{\mathbf{q}}, \vec{\mathbf{p}})\, |\vec{q}\rangle$$

The Hamiltonian will have some terms depending on $\vec{p}$ and others on $\vec{q}$. For a standard non-relativistic potential, these terms are separate, $H = \vec{\mathbf{p}}^2/2m + V(\vec{\mathbf{q}})$. The $\vec{\mathbf{q}}$ operators can act on the q-eigenstates, for which we have

$$\langle \vec{q}\,'|\, \vec{q}\rangle = \delta(\vec{q}\,' - \vec{q}), \qquad \langle \vec{q}\,'|\, V(\vec{\mathbf{q}})\, |q\rangle = \delta(\vec{q}\,' - \vec{q}) V(\vec{q}).$$

For the momentum-dependent pieces we introduce a complete set of momentum eigenstates,

$$1 = \int d\vec{p}\, |\vec{p}\rangle\, \langle \vec{p}|\,.$$

The normalization is such that

$$\langle \vec{q}\,|\, \vec{p}\rangle = \frac{e^{i\vec{p}\cdot\vec{q}/\hbar}}{(2\pi\hbar)^{D/2}},$$

where D is the number of degrees of freedom (number of q's). Then, for example,

$$\begin{aligned} \langle \vec{q}\,'|\, \vec{\mathbf{p}}^{\,2}\, |\vec{q}\rangle &= \int d\vec{p}\, \langle \vec{q}\,'|\, \vec{\mathbf{p}}^{\,2}\, |\vec{p}\rangle\, \langle \vec{p}\,|\, \vec{q}\rangle = \int d\vec{p}\, \vec{p}^{\,2}\, \langle \vec{q}\,'\,|\, \vec{p}\rangle \langle \vec{p}\,|\, \vec{q}\rangle \\ &= \int \frac{d\vec{p}}{(2\pi)^D} \vec{p}^{\,2} e^{i\vec{p}\cdot(\vec{q}\,' - \vec{q})}. \end{aligned}$$

By introducing the integral over $\vec{p}$, we are able to express the Hamiltonian in terms of c-numbers rather than operators. If the Hamiltonian has terms involving both q's and p's, there are questions of operator ordering. We will assume we get the right value if we replace $\vec{\mathbf{q}}$ by $(\vec{q}\,' + \vec{q})/2$ and $\vec{\mathbf{p}}$ by $\vec{p}$. The exact conditions are in Peskin.

Thus we see that we may replace the time-evolution operator matrix element

$$\langle \vec{q}\,'|\, e^{-iH\Delta t/\hbar}\, |\vec{q}\rangle \to \int \frac{d\vec{p}}{(2\pi)^D} e^{i\vec{p}\cdot(\vec{q}\,' - \vec{q})/\hbar} \exp\left[-iH\left(\frac{\vec{q}\,' + \vec{q}}{2}, \vec{p} \right) \frac{\Delta t}{\hbar} \right].$$

If we also use $\vec{q}\,' \approx \vec{q} + \dot{\vec{q}}\Delta t$, we have

$$\int \frac{d\vec{p}}{(2\pi)^D} \exp\left(\frac{i}{\hbar}\left[\vec{p}\cdot\dot{\vec{q}} - H\left(\frac{\vec{q}\,' + \vec{q}}{2}, \vec{p}\right)\right]\,\Delta t\right).$$

This certainly looks like the exponential is $i\Delta t/\hbar$ times the Lagrangian, and if we put all the slices together and combine the exponentials, the exponential becomes

$$\frac{i}{\hbar}\int_{t_i}^{t_f} dt\,\Big(\vec{p}(t)\cdot\dot{\vec{q}}(t) - H(\vec{q}(t), \vec{p}(t))\Big).$$

Then the integral looks like the action, and we can intepret the quantum mechanical time evolution operator

$$U(\vec{q}_f, \vec{q}_i; t_f, t_i) = \prod_{i=1}^{n-1} d\vec{q}_i \prod_{i=1}^{n} \frac{d\vec{p}_i}{(2\pi\hbar)^D} \exp\frac{i}{\hbar}\int_{t_i}^{t_f}\Big[\vec{p}\cdot\dot{\vec{q}} - H(\vec{q}, \vec{p})\Big]\,dt$$

as a sum over all paths of the phases $\exp iS/\hbar$, where $S = \int L\,dt$ is the action evaluated on that path. There is one objection, however: we need to integrate over the momenta, so that in $\vec{p}(t)\cdot\dot{\vec{q}}(t) - H(\vec{q}(t), \vec{p}(t))$, the usual constraint between the momenta and the coordinate time derivatives is not to be imposed. In the case were the Hamiltonian involves momenta only in the form $\sum p_i^2/2m_i$, the integral over the momenta is Gaussian and can be explicitly evaluated. When this is done, the momenta in the Hamiltonian do get replaced by the appropriate expression in $\dot{\vec{q}}$, defined as $[\vec{q}(t+\Delta t) - \vec{q}(t)]/\Delta t$, and the integral in the phase is actually the action.

The combination of integral measures above is given a pretty abbreviation:

$$\mathcal{D}q(t)\mathcal{D}p(t) \equiv \prod_{i=1}^{n-1} d\vec{q}_i \prod_{i=1}^{n} \frac{d\vec{p}_i}{(2\pi\hbar)^D},$$

and when the momenta integrals are done, the resulting measure is called, somewhat inconsistently,

$$\mathcal{D}q(t).$$

Note that in this form, each integral over $q(t)$ has a factor ($C^{-1}(\epsilon)$ in Peskin) which blows up as the spacing between integration times goes to zero.