Little Note on Fierz

What is $(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\zeta}$?

If $\alpha=\beta$, only $\sigma^0=\delta_{\alpha\beta}$ and $\sigma^3=\delta_{\alpha\beta}(-1)^{\alpha+1}$ contribute, giving

$$\delta_{\alpha\beta}\delta_{\gamma\zeta}\left(1-(-1)^{\alpha+\gamma}\right)=2\delta_{\alpha\beta}\delta_{\gamma\zeta}\left(1-\delta_{\alpha\gamma}\right)=2\epsilon_{\alpha\gamma}\epsilon_{\beta\zeta}\qquad\text{for }\alpha=\beta.$$

If $\alpha\neq\beta$, only σ^1 and σ^2 can contribute, in which case only $\gamma\neq\zeta$ gives nonzero, and then

- either $\alpha=\gamma$, and the two terms cancel, $\left(\sigma^1_{\alpha\beta}\right)^2+\left(\sigma^2_{\alpha\beta}\right)^2=0$,
- or $\alpha=\zeta$, $\beta=\gamma$ $\sigma^1_{\alpha\beta}\sigma^1_{\gamma\zeta}=1$, $\sigma^2_{\alpha\beta}\sigma^2_{\gamma\zeta}=1$, so we get -2.

Thus when $\alpha\neq\beta$, $(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\zeta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\zeta}$.

So in either case,

$$(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\zeta}=2\epsilon_{\alpha\gamma}\epsilon_{\beta\zeta}.$$

Another Approach

Consider the mapping of 2×2 matrices

$$M\to\sum_\mu\sigma^\mu M\sigma_\mu.$$

This is a linear real transformation, so we may describe it by its action on a basis of 2×2 matrices, in particular using the identity and the Pauli matrices:

$$\begin{aligned}
\mathbb{1} &\to \left(\sigma^0\right)^2-\sum_{j=1}^3\left(\sigma^j\right)^2=-2\,\mathbb{1}\\
\sigma^j &\to \sigma^0\sigma^j\sigma^0-\sum_{k=1}^3\sigma^k\sigma^j\sigma^k=\sigma^j+\sum_{k=1}^3\left[-2\delta_{jk}\sigma^k+\sigma^j\left(\sigma^k\right)^2\right]=(1-2+3)\sigma^j\\
&= 2\sigma^j
\end{aligned}$$

Thus

$$\sum_{\mu=0}^3 p_\mu\sigma^\mu\to-2\sum_{\mu=0}^3 p_\mu\sigma_\mu,$$

which is strange, but reverses σ_L and σ_R, in addition to multiplying by -2.

Now if we consider the right hand side of the Fierz identity as a transformation on $M_{\beta\gamma}$

$$M\to 2\epsilon M^T\epsilon,$$

$$\begin{aligned}
\mathbb{1} &\to -2\,\mathbb{1}\\
\sigma^j &\to 2(i\sigma_2)(\sigma^j)^T(i\sigma_2)=2\sigma^j
\end{aligned}$$

So, of course, these are the same transformation.

This latter approach is a useful one to take to consider the Fierz identity for Dirac matrices, rearranging the Dirac indices (a,b,c,d) on $\gamma^\mu_{ab}\gamma_{\mu\,cd}$.