The beta function
$$B(x, y)$$

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The Beta function is defined, for Re $\alpha > 0$, Re $\beta > 0$, as

$$B(\alpha, \beta) = \int_0^1 dx \, x^{\alpha - 1} (1 - x)^{\beta - 1}.$$

To evaluate this, consider

$$\begin{split} \Gamma(\alpha+\beta)B(\alpha,\beta) &= \int_0^\infty t^{\alpha+\beta-1}e^{-t}dt \int_0^1 dx \, x^{\alpha-1}(1-x)^{\beta-1} \\ &= \int_0^\infty t^{\beta-1}e^{-t}dt \int_0^t du \, u^{\alpha-1}(1-\frac{u}{t})^{\beta-1} \\ &= \int_0^\infty e^{-t}dt \int_0^t du \, u^{\alpha-1}(t-u)^{\beta-1} \\ &= \int_0^\infty du \, u^{\alpha-1} \int_u^\infty e^{-t}(t-u)^{\beta-1}dt \\ &= \int_0^\infty du \, u^{\alpha-1}e^{-u} \int_0^\infty e^{-v}v^{\beta-1}dv \\ &= \Gamma(\alpha)\Gamma(\beta), \end{split}$$

SO

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Other interesting forms:

By writing $x = \sin^2 \theta$ we have

$$B(\alpha, \beta) = 2 \int_0^{\pi/2} (\sin \theta)^{2\alpha - 1} (\cos \theta)^{2\beta - 1} d\theta.$$

By writing x = 1/(1+u) we have

$$B(\alpha, \beta) = \int_0^\infty \frac{u^{\alpha - 1}}{(1 + u)^{\alpha + \beta}} du.$$

 $\Gamma(z)$ has a pole of residue 1 at z=0, but it is important to have the finite term as well. For infinitesimal ϵ we have

$$\Gamma(1+\epsilon) = \int_0^\infty t^{\epsilon} e^{-t} dt \approx \int_0^\infty (1+\epsilon \ln t) e^{-t} dt = 1 - \epsilon \gamma,$$

where

$$\gamma = -\int_0^\infty (\ln t) e^{-t} dt = 0.57721\,56649\dots$$

is known as the Euler's constant or Mascheroni's constant and can also be written as

$$\gamma = \lim_{m \to \infty} \left(\sum_{j=1}^{m} \frac{1}{j} - \ln m \right).$$

Thus as $z \to 0$,

$$\Gamma(z) = z^{-1}\Gamma(1+z) = \frac{1}{z} - \gamma + \mathcal{O}(z).$$

There is lots more about Gamma functions in Whittaker and Watson, Modern Analysis, chapter 12.