Gauge Theory on a Lattice

One approach to field theory, in particular to aspects that are not well treated in perturbation theory, is to approximate the field defined on a space-time continuum with a lattice field theory, defined on a lattice of space-time points. There has been a great deal of cross-fertilization from considering renormalization from the continuum high-energy physics point of view and from the condensed matter, Wilsonian, viewpoint, in which the field theory only makes sense down to some small distance scale anyway. But we are going to consider field theory on a lattice for a different reason — because it helps to clarify the fundamental idea of gauge fields.

Symmetry

Consider a theory which involves a set of N real fields $\phi_i(x^\mu)$ which have an internal symmetry group[1] $\mathcal{G}$ under which they transform with a representation M, so that a particular symmetry transformation $G \in \mathcal{G}$ acts on the ϕ fields by

$$G : \phi_i(x) \mapsto \phi'_i(x) = \sum_j M_{ij}(G)\phi_j(x). \tag{1}$$

If it is a symmetry, the Lagrangian must be invariant. If the kinetic term is of the usual form, $\frac{1}{2}\sum_{\mu,i}\partial_\mu\phi_i\partial^\mu\phi_i$, invariance requires that M is an orthogonal matrix $\sum_k M_{ki}M_{kj} = \delta_{ij}$. That condition also insures the invariance of the mass term $-\frac{1}{2}\sum_i \phi_i^2$, and of any other "potential" term $V(\sum_i \phi_i^2)$ depending only on the "length" of ϕ. Provided V has that form, we see that the theory should be invariant under all the orthogonal transformations (1). We see that the individual components ϕ_i are only projections along the unit vectors of an arbitrary orthonormal basis of $\mathbb{R}^N$, and do not have separate intrinsic physical meanings. Alternatively, V might not be invariant under all of $\mathrm{O}(N)$, but only under the subgroup[2] $\mathcal{G}$. For example, one important group

[1] The notation is not completely standard. Many books would use G for the group, $\mathcal{G}$ for the Lie algebra of the group, and g for an element of G. Because we are going to use g as the analogue of the fundamental charge, I am using G for a group element, $\mathcal{G}$ for the group, and $\mathfrak{G}$ for the Lie algebra, elements of which will be called $\mathcal{A}$.

[2] More precisely, the image of $\mathcal{G}$ under the representation $M : \mathcal{G} \to N \times N$ matrices is a subgroup of $\mathrm{O}(n)$.

is the SU(3) of colors which act on each triplet (in color) of quarks. Replacing the 3 complex quark fields by 6 real fields, the kinetic term would be invariant under the group O(6) $\sim$ SU(4), but the interaction terms are only invariant under the subgroup SU(3).

So we are going to be considering a symmetry group[3] $\mathcal{G}$ which has generators L_b which form a basis of the "Lie algebra" $\mathfrak{G}$ of the group[4]. As we saw for the Lorentz group, the Lie algebra for SO(N) is the set of antisymmetric real $N \times N$ matrices, with $\frac{1}{2}N(N-1)$ independent generators $\tilde{L}$, or, for physicists, $\frac{1}{2}N(N-1)$ purely imaginary antisymmetric $N \times N$ matrices. For SU(3), the generators may be thought of as traceless hermitean 3×3 matrices.

Discretization

How might we approximate the continuum theory on a lattice? Instead of $\phi_i(\mathbf{x})$ defined for all values of $\mathbf{x} \in \mathbb{R}^4$, we might have $\phi_i(\vec{n})$ discrete variables defined only for integer values $\vec{n} \in \mathbb{Z}^4$, representing a hypercubic lattice in space-time with lattice spacing a, with $\mathbf{x}^\mu = an^\mu$. The mass term in the action

$$-\frac{1}{2}\int d^4x \sum_i \phi_i^2(\mathbf{x}) \to -\frac{1}{2}a^4 \sum_{\vec{n}\in\mathbb{Z}^4}\sum_i \phi_i^2(\vec{n}).$$

For the kinetic energy term we need to replace a derivative by a finite difference. The simplest substitution is to replace

$$\partial_\mu\phi_i(\mathbf{x}) \to \frac{1}{a}\left(\phi_i(\vec{n}+\vec{\Delta}_\mu) - \phi_i(\vec{n})\right),$$

where Δ_μ is 1 in the μ direction and 0 in the others. Here the relation of x^μ and $\vec{n}$ is $x^\nu = an^\nu + \frac{1}{2}a\delta^\nu_\mu$, representing most accurately the x in the middle

[3]We will only consider connected groups which are either Abelian or semisimple, or products of such groups.

[4]Here is what we will need to know about groups and Lie algebras: The algebra can be represented by generators L_a which satisfy $[L_a, L_b] = i\sum_k {c_{ab}}^k L_k$, with ${c_{ab}}^k$ real numbers known as the *structure constants* of the group. These give a bilinear *Killing form* β : $G \times G \to \mathbb{R}$ given by $\beta(L_i, L_j) = -\sum_{ab} {c_{ai}}^b {c_{bj}}^a$. As this is a real symmetric matrix, it can be diagonalized. For compact, semisimple groups, all the eigenvalues are positive, the L_i's can be scaled, so that $\beta_{ij} = 2\delta_{ij}$. Then the basis has be chosen such that $\sum_k L_k^2$ is a casimir operator, commuting with each of the L_a's, and it can be shown that the structure constants are totally antisymmetric. This should be familiar for the rotation group, and is explained in more detail in "Lightning review of group" and "Notes on Representations, the Adjoint rep, the Killing form, and antisymmetry of ${c_{ij}}^k$ ".

of the two lattice points. If we expand out the squares of the differences, we get terms which look just like the mass terms, but also nearest neighbor couplings $\sum_i \phi_i(\vec{n}+\Delta_\mu)\phi_i(\vec{n})$.

Each of these contributions to the action is still invariant under the transformation (1), providing we use the same group transformation at every point in space-time. This is called a global gauge transformation.

In a relativistic field theory, all information is local, because information can only travel at the speed of light. So we might ask, if the theory is unchanged by a group action at one point, why should that depend on having the same transformation at every other point? In other words, could we have a **local** symmetry, in which equation (1) holds with the group element varying from one point of space-time to another? The mass terms and other terms in $V(\phi)$ only depend on one point, so they don't care whether M varies, and they are invariant under such transformations. But the nearest-neighbor coupling

$$\sum_i \phi_i(\vec{n}+\Delta_\mu)\phi_i(\vec{n}) \to M_{ik}(G(\vec{n}+\Delta_\mu))M_{ij}(G(\vec{n}))\,\phi_k(\vec{n}+\Delta_\mu)\phi_j(\vec{n})$$

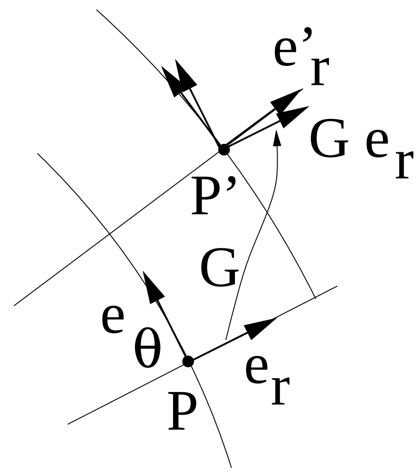
is not invariant because

$$M^{-1}(G(\vec{n}+\Delta_\mu))M(G(\vec{n})) \neq 1$$

if the G's (and hence the M's) vary from point to point.

Parallel Transport

The problem is that we have a term in the Lagrangian that is a function of how ϕ changes from point to point, but we measure that change by how much the components change, subtracting $\phi_j(x+\Delta)-\phi_j(x)$. That is only correct if the basis for comparing the ϕ's does not change. We must have a way to measure change from point to point, but before we can subtract one ϕ vector from another at a different point, we must "parallel transport" it to that new point. That is, for each link between neighboring points, we must have a rule for parallel transporting ϕ fields from one end of the link to the other. The change in the field $\boldsymbol{\phi}=\sum_a \phi^a\hat{e}_a$ as we go from point A to point B is equal to $\Delta\boldsymbol{\phi}=\sum_a(\phi^a_B-\phi^a_A)\hat{e}_a$ *only* if we can assume that the basis vectors don't change, $\hat{e}^A_a=\hat{e}^B_a$. If we allow for the possibility that the basis we have chosen at the point $\vec{n}+\Delta_\mu$ differs by a group element G from that which corresponds to parallel transport from $\vec{n}$, we get a more elaborate definition of $\Delta\boldsymbol{\phi}$.



As an example, it might help to think of an ordinary vector in the plane, expressed in polar coordinates. Consider the unit basis vectors $\vec{e}_r$ and $\vec{e}_\theta$ at the point P. If we transport $\vec{e}_r$ to the point P' while keeping it parallel to what it was, we arrive at the vector labelled Ge_r, which is not the same as the unit radial vector e'_r at the point P'.

Note that if we have a vector $\vec{V}' = V'_r\vec{e}\,'_r + V'_\theta\vec{e}\,'_\theta$ at P' which is unchanged from the vector $\vec{V} = V_r\vec{e}_r + V_\theta\vec{e}_\theta$ at P, we **do not have** $V'_r = V_r$.

Now in our example we had an *a priori* rule for what parallel transport means, but if we are to allow local gauge invariance, this rule becomes a new degree of freedom. This dynamical variable is actually one element of the symmetry group (and therefore perhaps several degrees of freedom, $\frac{1}{2}N(N-1)$ for $SO(N)$, the orthogonal transformations in N dimensions), for each point on the lattice and each direction we might parallel transport $\boldsymbol{\phi}$. We can then build a theory with a local symmetry, but at the expense of introducing a lot of new degrees of freedom.

The theory that emerges from these consideration is a **gauge field theory**. Its degrees of freedom include not only the "matter fields" at each site of the lattice, but also "gauge fields" on each link between nearest neighbors. The matter fields live in a vector space which transforms linearly as a representation[5] of the "gauge group" $\mathcal{G}$. The gauge fields live in the group itself, at least in the lattice field theory, but may alternately be considered to take values in the Lie algebra of generators of the group, especially if we are to take the continuum limit of the lattice.

Covariant Derivative

When a group element G acts on a vector $\vec{V} = \sum_i V_i\hat{e}_i$ which transforms under a representation M, the components of the new vector are multiplied

[5] To a physicist, the vector space in which the matter fields live is called the representation, but to mathematicians the representation consists of the matrices M_{ij}, or more accurately the mapping from elements of the group into matrices, $G \mapsto M(G)$.

by the matrix:

$$G : \vec{V} \to \vec{V}\,' = \sum_{ij} M_{ij}(G) V_j \hat{e}_i, \quad \text{so } V_i' = \sum_j M_{ij}(G) V_j.$$

So if G parallel transports $\vec{\phi}(\vec{n})$ from $\vec{n}$ to $\vec{n} + \vec{\Delta}_\mu$, and if we subtract this from $\vec{\phi}(\vec{n} + \Delta_\mu)$ to get the change in ϕ, we have

$$\Delta\phi = \sum_i \left[\phi_i(\vec{n} + \Delta_\mu) - \sum_j M_{ij}(G)\phi_j(\vec{n}) \right] \hat{e}_i.$$

If the fields are slowly varying over the distance of one lattice spacing, which is necessary if we are to consider the lattice an approximation to the continuum, we can approximate

$$\phi_i(\vec{n} + \Delta_\mu) \approx \phi_i(\vec{n}) + a\partial_\mu\phi_i.$$

We can also assume that the group transformation that parallel transports by one lattice spacing is close to the identity, and that the Lie algebra element which generates it should be proportional to the lattice spacing a. Thus we may write $G = e^{iag\mathcal{A}}$, $M(G) = M(e^{iag\mathcal{A}}) \approx 1 + iagM(\mathcal{A})$, where $\mathcal{A}$ is an element in the Lie algebra $\mathfrak{G}$ of the gauge group $\mathcal{G}$. [We have added a parameter g which will turn out to be the fundamental charge, in order to get conventionally defined $\mathcal{A}$ fields, although sometimes that is not done, and the scale for measuring $\mathcal{A}$ is the natural one for the group.] Then we find, to first order in the lattice spacing a,

$$\Delta\phi_i = a \left(\partial_\mu\phi_i - ig \sum_j M_{ij}(\mathcal{A})\phi_j \right).$$

In the continuum limit, we define $1/a$ times this to be the **covariant derivative**, but first I must say a few words about the gauge field $\mathcal{A}$. First, as there is a different value on each link, and in the continuum limit there are four[6] links radiating from each point, we need to be defining four fields $\mathcal{A}_\mu(\mathbf{x})$. Also, each $\mathcal{A}_\mu$ is not a single field, in general, but an element of the Lie algebra, which is a vector space. The Lie algebra for the 3-D rotation group, for

[6]Actually there are eight, as there are forward and backwards links in each direction. But the "backwards" ones can be thought of as belonging to "previous" sites.

example, is parameterized by a vector with three components, $\vec{\omega}$. Rotations themselves are not a gauge group, but one possible gauge group to consider is the $SU(2)$ of the electro-weak theory, which is isomorphic[7] to the rotation group. One usually uses L_i to represent a basis vector of the Lie algebra vector space, so the gauge field can be expanded as

$$\mathcal{A}_\mu(\mathbf{x}) = \sum_b A_\mu^{(b)}(\mathbf{x}) L_b.$$

This brings us to the definition of the **covariant derivative**:

$$(D_\mu\phi)_j = \partial_\mu\phi_j - ig\sum_{kb} A_\mu^{(b)} M_{jk}(L_b)\phi_k\,; \qquad D_\mu\phi = \partial_\mu\phi - igA_\mu^{(b)}M(L_b)\phi,$$

where on the right we have written the expression with implied summations on matrix and Lie algebra indices and implied multiplication.

Gauge Transformations

What does this have to do with local symmetry? We saw that the transformation (1), where we let G vary with x, is a symmetry for the lattice terms involving only a single site, but *not* for the kinetic term, $(\partial\phi)^2$, which involves cross terms such as $\sum_i \phi_i(\vec{n}+\Delta_\mu)\phi_i(\vec{n})$. These couple neighboring points, and are not invariant. But with our improved definition of $(\Delta\phi)$, the cross terms now have the form

$$\phi(\vec{n}+\Delta_\mu)\cdot M(G_L)\cdot\phi(\vec{n}),$$

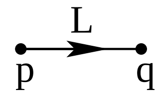
where G_L is the group transformation associated with the link $(\vec{n},\vec{n}+\Delta_\mu)$.

We can now ask what happens under the transformation in a different way. If we think of the gauge transformation $G(\mathbf{x})$ in the passive language as a change in the basis elements for the matter fields, we realize that it will also effect the rule for doing parallel transport. If G_L was the group transformation on the basis which did a parallel transport from site p to site q, with link L going from p to q, then after a change of basis by G_p at p and one by G_q at q, the way to parallel transport the new basis at p must be $G'_L = G_q G_L G_p^{-1}$. So we now define the gauge transformation Λ, which is

[7] Not exactly: the Lie algebra of $SU(2)$ is the same as the Lie algebra of the three dimensional rotation group $SO(3)$, but the actual groups differ, as is discussed when considering how spinors transform under rotations of 2π.

specified by a group element at each lattice site

$$\Lambda : \begin{cases} \phi(x_p) \to M(G_p) \cdot \phi(x_p) \\ \phi(x_q) \to M(G_q) \cdot \phi(x_q) \\ \quad G_L \to G_q G_L G_p^{-1} \end{cases}$$



This gauge transformation is a **local symmetry** of the gauge field theory. Let's verify that this is an invariance of the nearest neighbor term:

$$\begin{aligned} \phi(x_q) \cdot M(G_L) \cdot \phi(x_p) &= \phi_i(x_q) M_{ij}(G_L) \phi_j(x_p) \\ &\to M_{ik}(G_q)\phi_k(x_q) M_{ij}(G_q G_L G_p^{-1}) M_{j\ell}(G_p)\phi_\ell(x_p) \\ &\quad = \phi_k(x_q) M^{-1}_{ki}(G_q) M_{ij}(G_q G_L G_p^{-1}) M_{j\ell}(G_p)\phi_\ell(x_p) \\ &\quad = \phi_k(x_q) M_{k\ell}(G_L)\phi_\ell(x_p) = \phi(x_q)\cdot M(G_L)\cdot\phi(x_p), \end{aligned}$$

where we have used the orthogonality of $M(G_q)$ and the fact that the M's are a representation, and therefore $M^{-1}_{ki}(G_q) M_{ij}(G_q G_L G_p^{-1}) M_{j\ell}(G_p) = M_{k\ell}(G_L)$.

In a continuum field theory, we consider only local gauge transformations where the group element varies differentially in the continuum limit. We may think of Λ as given by a Lie-algebra valued scalar field $\lambda(\mathbf{x}) = \sum_b \lambda^{(b)}(\mathbf{x}) L_b$. Then the matter fields transform as

$$\phi(\mathbf{x}) \to \phi'(\mathbf{x}) = e^{i\sum_b \lambda^{(b)}(\mathbf{x}) M(L_b)}\phi(\mathbf{x}),$$

while the gauge field itself transforms by

$$A^{(b)}_\mu(\mathbf{x}) \to A'^{(b)}_\mu(\mathbf{x}),$$

with

$$e^{iagA'^{(b)}_\mu(\mathbf{x})} = e^{i\lambda(\mathbf{x}+\frac{1}{2}a\Delta_\mu)} e^{iagA^{(b)}_\mu(\mathbf{x})L_b} e^{-i\lambda(\mathbf{x}-\frac{1}{2}a\Delta_\mu)}. \tag{2}$$

We have placed x at the middle of the link. We now expand to first order in the lattice spacing, remembering that $\lambda(\mathbf{x})$ and $\partial_\mu\lambda(\mathbf{x})$ may not commute. So we will expand the exponential rather than λ. Approximating

$$\begin{aligned} e^{iag\mathcal{A}_\mu} &\to 1 + iag\mathcal{A}_\mu, \\ e^{i\lambda(\mathbf{x}\pm\frac{1}{2}a\Delta\mu)} &\to e^{i\lambda(\mathbf{x})} \pm \frac{1}{2}a\partial_\mu\left[e^{i\lambda(\mathbf{x})}\right], \end{aligned}$$

and plugging these into (2), we get

$$\begin{aligned} 1 + iag\mathcal{A}'_\mu &= \left(e^{i\lambda} + \frac{1}{2}a\partial_\mu e^{i\lambda}\right)(1 + iag\mathcal{A}_\mu)\left(e^{-i\lambda} - \frac{1}{2}a\partial_\mu e^{-i\lambda}\right) \\ &= 1 + iage^{i\lambda}\mathcal{A}_\mu e^{-i\lambda} + \frac{1}{2}a\left(\partial_\mu e^{i\lambda}\right)e^{-i\lambda} - \frac{1}{2}ae^{i\lambda}\left(\partial_\mu e^{-i\lambda}\right) \end{aligned}$$

Note from $\partial_\mu\left(e^{i\lambda}e^{-i\lambda}\right) = 0$ that the third and fourth terms are equal, so we can drop the third and double the fourth, to get

$$\begin{aligned}\mathcal{A}'_\mu &= e^{i\lambda}\mathcal{A}_\mu e^{-i\lambda} + \frac{i}{g}e^{i\lambda}\partial_\mu e^{-i\lambda} \\ &= e^{i\lambda}\left(\mathcal{A}_\mu + \frac{i}{g}\partial_\mu\right)e^{-i\lambda}\end{aligned}$$

Let us now ask how this is related to the gauge transformations we know from Maxwell's theory, which look less complicated. Electromagnetism is a gauge field, but one with a very simple gauge group, that of rotations about a single fixed axis[8]. The group consists of $\mathcal{G} = \{e^{i\theta L_1}\}$ and the Lie algebra has only one generator, L_1, and is therefore isomorphic to the real line $\mathbb{R}$, and the single structure constant ${c_{11}}^1$ is zero (a counterexample to assuming that the Killing form can always be set to $2 \times 1\!\!1$). The rotations act on charged fields, which are usually represented by complex fields Φ but in our treatment here are represented by a doublet of real fields, $(\phi_1, \phi_2) = (\text{ Re }\Phi, \text{ Im }\Phi)$. The transformation

$$\phi \to \phi' = \begin{pmatrix}\text{Re }\Phi' \\ \text{Im }\Phi'\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\text{Re }\Phi \\ \text{Im }\Phi\end{pmatrix}$$

gives $\Phi' = e^{i\theta}\Phi$, so the gauge transformations are **local** changes in phase of the charged fields. The gauge transformations of fields themselves is vastly simplified by the fact that all the terms commute, so

$$A'_\mu = e^{i\lambda}\left(A_\mu + \frac{i}{g}\partial_\mu\right)e^{-i\lambda} = A_\mu + g^{-1}\partial_\mu\lambda.$$

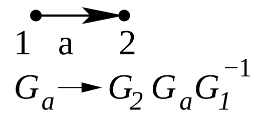
But this simplicity only holds for an **Abelian** group, one where all the generators commute, which is not enough when we wish to consider the gauge theories of the electroweak and strong interactions.

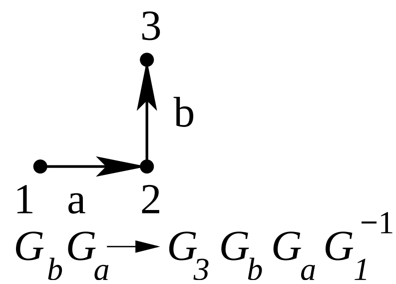
Pure Gauge Terms in $\mathcal{L}$

We now know how the kinetic terms for charged fields are modified by the presence of an external gauge field, with $\partial_\mu \to D_\mu$, but we have not yet

[8]These are not rotations in real space, but in some abstract space of field configurations. For QED that abstract space was represented by complex numbers, and the rotation is simply multiplication by $e^{i\theta}$ for a real phase θ.

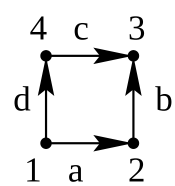
discussed the terms which propagate the gauge fields themselves. We need these terms in the Lagrangian to be invariant under gauge transformations. In particular this means that they cannot depend only on a single link, because we can always make a gauge transformation $G_1 = G_a$ which resets the group element for a single link to 1, so there would be no dependence on the field. In fact, the simplest way to get rid of the gauge dependance of $G_a = e^{iag\mathcal{A}_x(\mathbf{x}_a)}$ on $G_2 = e^{i\lambda(\mathbf{x}_2)}$ is to premultiply it by G_b,

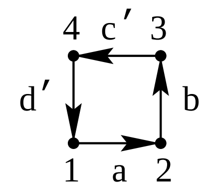




$$G_b G_a \to G_3 G_b G_2^{-1} G_2 G_a G_1^{-1} = G_3 G_b G_a G_1^{-1}.$$

There is still a gauge dependence on the endpoints of the path, however, so the best thing to do is close the path. To do so, we are traversing some links backwards from the way they were defined, but from that definition in terms of parallel transport it is clear that the group element associated with taking a link backwards is the inverse of the element taken going forwards. So the group element associated with the closed path on the right (which is called a plaquette) is $G_P = G_d^{-1} G_c^{-1} G_b G_a$, which transforms under gauge transformations as



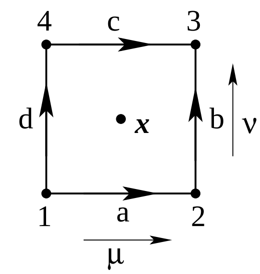


$$\begin{aligned}
G_P \to G'_P &= \left(G_4 G_d G_1^{-1}\right)^{-1} \left(G_3 G_c G_4^{-1}\right)^{-1} G_3 G_b G_2^{-1} G_2 G_a G_1^{-1} \\
&= G_1 G_d^{-1} G_4^{-1} G_4 G_c^{-1} G_3^{-1} G_3 G_b G_2^{-1} G_2 G_a G_1^{-1} \\
&= G_1 G_d^{-1} G_c^{-1} G_b G_a G_1^{-1} \\
&= G_1 G_P G_1^{-1}.
\end{aligned}$$

So the plaquette group element is not invariant but it does have a simpler and more restricted variation. In the continuum limit we expect each link's group element to be near the identity and also to have G_c differ from G_a by something proportional to the lattice spacing, so G_P should be close to the identity, the difference considered a generator in the Lie algebra. The Killing form acting on that generator will provide us with an invariant. Let us define

$\mathcal{F}_{\mu\nu} = -ia^{-2}g^{-1}(G_P - 1)$ to be the field-strength tensor, where μ and ν are the directions of links a and b respectively. Let us take $\mathbf{x}$ in the center of the placquette. Expanding each link to order $\mathcal{O}(a^2)$

$$\begin{aligned} G_a &\approx 1 + iag\mathcal{A}_\mu(\mathbf{x} - \frac{1}{2}a\Delta_\nu) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x} - \frac{1}{2}a\Delta_\nu) \\ &\approx 1 + iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x}) \\ G_c^{-1} &\approx 1 - iag\mathcal{A}_\mu(\mathbf{x} + \frac{1}{2}a\Delta_\nu) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x} + \frac{1}{2}a\Delta_\nu) \\ &\approx 1 - iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x}), \end{aligned}$$



we have, to second order in a[9],

$$\begin{aligned} G_P &= \left(1 - iag\mathcal{A}_\nu(\mathbf{x}) + \frac{1}{2}ia^2g\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\nu^2(\mathbf{x})\right) \\ &\quad \left(1 - iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x})\right) \\ &\quad \left(1 + iag\mathcal{A}_\nu(\mathbf{x}) + \frac{1}{2}ia^2g\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\nu^2(\mathbf{x})\right) \\ &\quad \left(1 + iag\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}ia^2g\partial_\nu\mathcal{A}_\mu(\mathbf{x}) - \frac{1}{2}a^2g^2\mathcal{A}_\mu^2(\mathbf{x})\right) \\ &= 1 + a^2g\left\{g\left[\mathcal{A}_\mu(\mathbf{x}), \mathcal{A}_\nu(\mathbf{x})\right] + i\partial_\mu\mathcal{A}_\nu(\mathbf{x}) - i\partial_\nu\mathcal{A}_\mu(\mathbf{x})\right\} \end{aligned}$$

Thus

$$\mathcal{F}_{\mu\nu}(\mathbf{x}) = \partial_\mu\mathcal{A}_\nu(\mathbf{x}) - \partial_\nu\mathcal{A}_\mu(\mathbf{x}) - ig\left[\mathcal{A}_\mu(\mathbf{x}), \mathcal{A}_\nu(\mathbf{x})\right].$$

Note that $\mathcal{F}_{\mu\nu}$ is

- a Lie-algebra valued field, $\mathcal{F}_{\mu\nu}(\mathbf{x}) = \sum_b F_{\mu\nu}^{(b)}(\mathbf{x})L_b$, because $[,]$ is closed in a Lie algebra.
- An antisymmetric tensor, $\mathcal{F}_{\mu\nu}(\mathbf{x}) = -\mathcal{F}_{\nu\mu}(\mathbf{x})$.
- Because the Lie algebra is defined in terms of the structure constants, ${c_{ab}}^d$ by

$$[L_a, L_b] = i{c_{ab}}^d L_d,$$

[9] Note that the terms in $\mathcal{A}_\mu^2(\mathbf{x})$ cancel, and only the commutator, not the product, of L_a's is left.

the field-strength tensor may also be written

$$F_{\mu\nu}^{(d)} = \partial_\mu A_\nu^{(d)} - \partial_\nu A_\mu^{(d)} + g c_{ab}{}^d A_\mu^{(a)} A_\nu^{(b)}.$$

Before we turn to the Lagrangian, let me point out a crucial relationship between the covariant derivatives and the field-strength. If we take the commutator of covariant derivatives

$$D_\mu = \partial_\mu - igA_\mu^{(b)} L_b$$

at the same point but in different directions,

$$\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu - ig\mathcal{A}_\mu, \partial_\nu - ig\mathcal{A}_\nu] = -ig\partial_\mu\mathcal{A}_\nu - g^2\mathcal{A}_\mu\mathcal{A}_\nu - (\mu \leftrightarrow \nu) \\
&= -g^2 [\mathcal{A}_\mu, \mathcal{A}_\nu] - ig\partial_\mu\mathcal{A}_\nu + ig\partial_\nu\mathcal{A}_\mu \\
&= -ig\mathcal{F}_{\mu\nu}.
\end{aligned}$$

Notice that although the covariant derivative is in part a differential operator, the commutator has neither first or second derivatives left over to act on whatever appears to the right. It does need to be interpreted, however, as specifying a representation matrix that will act on whatever is to the right.

Now consider adding to the Lagrangian a term proportional to the Killing form evaluated on $\mathcal{F}$, twice, $\beta(\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}) = 2\sum_b F_{\mu\nu}^{(b)} F^{(b)\,\mu\nu}$. I have assumed the generators L_a have been normalized so that the Killing form $\beta(L_a, L_b) = 2\delta_{ab}$, and the stucture constants are totally antisymmetric[10]. We know that under a gauge transformation $\mathcal{F}_{\mu\nu} \to e^{i\lambda}\mathcal{F}_{\mu\nu}e^{-i\lambda}$. If λ is infinitesimal, $\mathcal{F}_{\mu\nu} \to \mathcal{F}_{\mu\nu} + i[\lambda, \mathcal{F}_{\mu\nu}] = \left\{ F_{\mu\nu}^{(d)} - \lambda^{(a)} F_{\mu\nu}^{(b)} c_{ab}{}^d \right\} L_d$, so

$$\delta\beta(\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}) = 2\beta(\delta\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}) = 2 \times (-2)\lambda^{(a)} F_{\mu\nu}^{(b)} c_{ab}{}^d F^{(d)\,\mu\nu} = 0$$

where the expression vanishes because $c_{ab}{}^d$ is antisymmetric under interchange of b and d but $F_{\mu\nu}^{(b)} F^{(d)\,\mu\nu}$ is symmetric under the same interchange (and we are summing on b and d). As $\beta(\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu})$ doesn't change to first order under infinitesimal transformations, it also doesn't change under the finite transformations they generate.

[10] See `groups.pdf` and `adjnote.pdf` on the Lecture and Supplementary Notes page of the course website.

Equations of Motion for the Gauge Fields

We choose the normalization of the A fields so that the pure gauge term in the Lagrangian density is $-\frac{1}{4}F^{(b)}_{\mu\nu}F^{(b)\,\mu\nu}$. Suppose we also have Dirac matter fields transforming under a representation $t^b_{ij} = M_{ij}(L_b)$ of the group, and perhaps some scalar fields as well, transforming under a (possibly) different representation $\bar{t}^b_{ij} = \bar{M}_{ij}(L_b)$, where the bars here only represent a different representation, not any kind of conjugation. The gauge fields come into the matter terms in the Lagrangian because, in order to maintain local gauge invariance, all derivatives need to be replaced by covariant derivatives. Thus the potential terms for matter fields in $\mathcal{L}$ will not be involved in the equations of motion of the gauge fields, and we need only look at

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{(b)}_{\mu\nu}F^{(b)\,\mu\nu} + i\bar{\psi}\gamma^\mu\left(\partial_\mu - igA^{(b)}_\mu t^b\right)\psi \\ &\quad+\frac{1}{2}\left[\left(\partial_\mu - igA^{(b)}_\mu \bar{t}^b\right)\phi\right]^T\left[\left(\partial^\mu - igA^{(b)\,\mu}\bar{t}^b\right)\phi\right].\end{aligned}$$

Let us analyze the classical mechanics of the gauge fields. First we find the canonical momentum conjugate to $A^{(b)}_\mu(\mathbf{x})$, which is

$$\pi^{(b)\,\mu}(\mathbf{x}) = \frac{\delta\mathcal{L}}{\delta\dot{A}^{(b)}_\mu}(\mathbf{x}) = -\frac{1}{2}F^{(b)\,\rho\nu}\frac{\delta F^{(b)}_{\rho\nu}}{\delta\dot{A}^{(b)}_\mu} = -F^{(b)\,0\mu}(\mathbf{x}).$$

Notice that

- The matter field terms do not contribute to the canonical momenta, because they depend on $\mathcal{A}_\mu$ but not its time derivative.
- The momentum conjugate to $\mathcal{A}_0$ is **identically** zero.

Momenta are not supposed to be identically zero in ordinary classical mechanics, they are supposed to be substitutes for velocities, *i.e.* time derivatives of the coordinates. The N coordinates and N momenta of an N degrees-of-freedom problem are supposed to span a $2N$ dimensional phase space. $\pi^0 = 0$ is not an equation of motion, it is a constraint. To properly handle such a situation we would need either to eliminate the constrained degrees of freedom or use some fancy techniques not usually discussed in classical mechanics courses. The canonical form of quantum mechanics would seem to require zero to not commute with $\mathcal{A}_0$, which is too strange to contemplate even in quantum mechanics. But it turns out that this situation can be handled somewhat straightforwardly in the path integral formulation of quantum mechanics.

In the classical mechanics of a field theory we generally define a generalization of the momentum,

$$\Pi_i^\nu(\mathbf{x}) = \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_i}, \qquad \Pi_i^0(\mathbf{x}) = \pi_i(\mathbf{x})$$

for each field ϕ_i. The $\nu = 0$ component is the conjugate momentum, but all components enter the Euler-Lagrange equations

$$\partial_\nu\Pi^\nu = \frac{\delta\mathcal{L}}{\delta\phi}.$$

Here our fields are not scalars but have additional indices b and μ because $\mathcal{A}$ is both Lie-algebra-valued and a vector. So the Euler-Lagrange equation is

$$\partial_\nu\Pi^{(b)\,\mu;\nu} = \frac{\delta\mathcal{L}}{\delta A_\mu^{(b)}},$$

where

$$\Pi^{(b)\,\mu;\nu} = \frac{\delta\mathcal{L}}{\delta\left(\partial_\nu A_\mu^{(b)}\right)} = -F^{(b)\,\nu\mu}.$$

Let's turn to evaluating the right-hand side of the Euler-Lagrange equation, $\delta\mathcal{L}/\delta A_\mu^{(b)}$. Recall that the derivative intended here is to look for terms depending directly on $A_\mu^{(b)}$ considering the derivatives fixed. We will need to differentiate the field-strength

$$F_{\mu\nu}^{(c)} = \partial_\mu A_\nu^{(c)} - \partial_\nu A_\mu^{(c)} + g c_{ab}{}^c A_\mu^{(a)} A_\nu^{(b)},$$

but only the last term depends on undifferentiated gauge fields, so

$$\frac{\delta F_{\mu\nu}^{(c)}}{\delta A_\rho^{(b)}} = g c_{ab}{}^c \left(A_\mu^{(a)}\delta_\nu^\rho - A_\nu^{(a)}\delta_\mu^\rho\right),$$

and thus

$$\frac{\delta}{\delta A_\rho^{(b)}}\left[-\frac{1}{4}F_{\mu\nu}^{(c)}F^{(c)\,\mu\nu}\right] = -g c_{ab}{}^c F^{(c)\,\mu\rho} A_\mu^{(a)}.$$

The contributions to $\delta\mathcal{L}/\delta A_\rho^{(b)}$ from the matter terms are more straightforward — each D_ρ will contribute a $-igt^b$ or equivalent, so

$$\frac{\delta}{\delta A_\rho^{(b)}}\left\{+i\bar\psi\gamma^\mu D_\mu\psi + \frac{1}{2}\left[D_\mu\phi\right]^T D_\mu\phi\right]\right\} = g\bar\psi\gamma^\rho t^b\psi + ig\phi^T\bar{t}^b D^\rho\phi,$$

where I have used the fact that the representations of the generators $\bar{t}$ for real unitary representations are antisymmetric imaginary matrices.

Define the current

$$j^a_\nu = \bar{\psi}\gamma_\nu t^a \psi + i\phi^T \bar{t}^a D_\nu \phi,$$

so that we have found

$$\partial_\nu F^{(b)\,\nu\mu} - g c_{ab}{}^c F^{(c)\,\nu\mu} A^{(a)}_\nu + g j^{b\,\mu} = 0. \tag{3}$$

The appearence of the first two terms is reminiscent of a covariant derivative, but we have not explicitly defined the covariant derivative acting on Lie-algebra valued fields, because we have not explicitly explored their representation under the group. While the local gauge transformation of the gauge field is more complicated than can be described by a representation of the gauge group, under a *global* gauge transformation it or any other Lie-algebra valued field transforms as

$$e^{i\lambda^c L_c} : \mathcal{A} \to e^{i\lambda^c L_c} \mathcal{A} e^{-i\lambda^c L_c} = M^{\rm adj}_{ab}\left(e^{i\lambda^c L_c}\right) A^b L_a,$$

which is the adjoint representation. Differentiating wrt λ_c and setting λ to zero gives

$$[L_c, A^b L_b] = M^{\rm adj}_{ab}\left(L_c\right) A^b L_a = i c_{cb}{}^a A^b L_a,$$

so the appropriate representation for the covariant derivative of a Lie-algebra valued field is

$$M^{\rm adj}_{ab}\left(L_c\right) = i c_{cb}{}^a.$$

$M^{\rm adj}_{ab}\left(L_c\right)$ is called the **adjoint** representation, and is always of the same dimension as the Lie algebra itself.

As discussed above[11] we have normalized our generators L_b so that the structure constants are totally antisymmetric, and we can substitute

$$i M^{\rm adj}_{ab}\left(L_c\right) = -c_{cb}{}^a = c_{ab}{}^c$$

in the expression (3), giving

$$\partial_\rho F^{(a)\,\rho\mu} - i g A^{(c)}_\rho M^{\rm adj}_{ab}(L_c) F^{(b)\,\rho\mu} + g j^{a\,\mu} = D_\rho F^{(a)\,\rho\mu} + g j^{a\,\mu} = 0.$$

[11] And more extensively in my notes on "Adjoint Representation, Killing forms and the antisymmetry of $c_{ij}{}^k$", in the Supplementary Notes on the course website.

Now that we have defined the covariant derivative of a Lie algebra valued field,

$$D_\mu \lambda^{(b)} = \partial_\mu \lambda^{(b)} - ig A_\mu^{(c)} M_{bd}^{\mathrm{adj}}(L_c) \lambda^{(d)},$$

we may note that for an infinitesimal gauge transformation,

$$A_\mu^{(b)} \to A_\mu'^{(b)} = A_\mu^{(b)} + i\lambda^d A_\mu^{(c)} (i{c_{dc}}^b) + \frac{1}{g} \partial_\mu \lambda^{(b)} = A_\mu^{(b)} + \frac{1}{g} \left(D_\mu \lambda \right)^{(b)} .$$

The theory we have just defined, the gauge theory based on a non-Abelian Lie group, is known as Yang-Mills theory.