Functional Integral for a Scalar Field.

We have seen that the time evolution operator in quantum mechanics can be understood as a functional integral

$$\langle q_f, t_f | \, e^{-iH[t_f - t_i]} \, | q_i, t_i \rangle = \int \mathcal{D}q\mathcal{D}p \exp i \int dt \, [p\dot{q} - H(q,p,t)] \, ,$$

where from now on $\hbar = 1$. This is quite general, so we are not restricted to a finite number of degrees of freedom q. In particular, we could consider a scalar field with Hamiltonian

$$H = \int d^3\vec{x} \left\{ \frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}(\nabla\phi)^2(\vec{x}) + V(\phi(\vec{x})) \right\} .$$

[For a free scalar field of mass m, $V(\phi) = \frac{1}{2}m^2\phi^2$.] Then we need

$$\int \mathcal{D}\phi\mathcal{D}\pi \exp i \int d^3\vec{x}dt \left[-\frac{1}{2}\pi^2(x^\mu) + \pi(x^\mu)\dot{\phi}(x^\mu) \right] \exp i \int d^3\vec{x}dt \left[-\frac{1}{2}(\nabla\phi)^2 - V(\phi) \right].$$

Only the first exponential depends on $\pi(x^\mu)$, and we can treat each point of space-time independently. Using

$$\int \frac{dx}{\sqrt{2\pi}} \exp \left[-\frac{1}{2}\lambda x^2 + Bx \right] = \frac{e^{B^2/2\lambda}}{\sqrt{\lambda}},$$

with $\lambda = i\Delta^4 x$, and ignoring the field-independent normalization factors, we have

$$\int \mathcal{D}\pi \, \exp i \int d^3\vec{x}dt \left[-\frac{1}{2}\pi^2(x^\mu) + \pi(x^\mu)\dot{\phi}(x^\mu) \right] \sim \exp \, i \int d^3xdt \, \frac{1}{2}\dot{\phi}^2,$$

and thus the functional integral becomes

$$\int \mathcal{D}\phi \exp i \int d^4x \left[\frac{1}{2}\dot{\phi}^2(x^\mu) - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right] = \int \mathcal{D}\phi \; e^{i \int d^4x \mathcal{L}(\phi)},$$

where the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi).$$

We now have a form of quantum mechanics based on the Lagrangian rather than the Hamiltonian, and except for the integration region (which goes from t_i to t_f in time but includes all of space) our formulation is Lorentz invariant. It will also have symmetries under any transformation which leaves the Lagrangian density $\mathcal{L}$ invariant.

Correlation Functions

In quantum mechanics, the expectation values or transitions of operators are given by $\langle\psi_f|\,\mathcal{O}_1\mathcal{O}_2\,|\psi_i\rangle$, where we generally think of operators at a fixed time. But in field theory we need to think of operators acting at arbitrary points in space-time. In the Schrödinger picture, where the fundamental operators are considered time-independent but the states evolve, the evolution $\phi_i \to \phi_f$ under the influence of some operators $\mathcal{O}'(x_2^\mu)\mathcal{O}(x_1^\mu)$ is given by

$$\langle\phi_f|\, e^{-i(t_f-x_2^0)H}\mathcal{O}'_S(x_2^\mu)e^{-i(x_2^0-x_1^0)H}\mathcal{O}_S(x_1^\mu)e^{-i(x_1^0-t_i)H}\,|\phi_i\rangle\,,$$

where I have assumed x_2 is later than x_1. We may rewrite this in the Heisenberg picture, in which operators evolve while the states are fixed. The connection is

$$\mathcal{O}_H(\vec{x},t) = e^{iHt}\mathcal{O}_S(\vec{x})e^{-iHt}.$$

Then the transition amplitude is

$$\langle\phi_f|\, e^{-it_fH}\mathcal{O}'_H(x_2^\mu)\mathcal{O}_H(x_1^\mu)e^{it_iH}\,|\phi_i\rangle\,.$$

Consider the operators to be the fields themselves. Returning to the Schrödinger picture, we see that

$$\begin{aligned}
&\langle\phi_f|\, e^{-it_fH}\phi_H(x_2^\mu)\phi_H(x_1^\mu)e^{it_iH}\,|\phi_i\rangle\\
&= \langle\phi_f|\, e^{-i(t_f-x_2^0)H}\phi_S(x_2^\mu)e^{-i(x_2^0-x_1^0)H}\phi_S(x_1^\mu)e^{-i(x_1^0-t_i)H}\,|\phi_i\rangle\\
&= \int_{t_2}\mathcal{D}\phi_2(\vec{x})\,U(\phi_f,\phi_2,t_f,t_2)\phi_2(x_2^\mu)\int_{t_1}\mathcal{D}\phi_1(\vec{x})\,U(\phi_2,\phi_1,t_2,t_1)\phi_1(x_1^\mu)U(\phi_1,\phi_i,t_1,t_i)\\
&= \int\mathcal{D}\phi\;\phi(x_2^\mu)\phi(x_1^\mu)\exp\;i\int_{t_i}^{t_f}d^4x\mathcal{L}(\phi).
\end{aligned}$$

Note that the $\int\mathcal{D}\phi$ in the next-to-last line are integrals only over states at a given time, while the $\int\mathcal{D}\phi$ in the last line is a functional integral over space-time.

The connection of the operator expectation value to the functional integral assumed $x_2^0 > x_1^0$. The functional integral itself is unchanged under $x_1 \leftrightarrow x_2$, as the ϕ's there are commuting c-numbers. But in the operator form $\phi_H(x_2^\mu)$ and $\phi_H(x_1^\mu)$ need not commute, and our expression is only correct if the later one occurs first. We therefore introduce the "Time ordering operator" T, which tells us to order any operators appearing in its scope in decreasing order of their times. This is not an operator on the Hilbert space;

rather it is a metaoperator which acts on our symbolic expressions, reordering them before they are allowed to physically act on the Hilbert space. Our corrected expression is thus

$$\langle\phi_f| \, e^{-it_f H} T\left(\phi_H(x_2^\mu)\phi_H(x_1^\mu)\right) e^{it_i H} \, |\phi_i\rangle = \int \mathcal{D}\phi \; \phi(x_2^\mu)\phi(x_1^\mu) \exp \; i \int_{t_i}^{t_f} d^4x \mathcal{L}(\phi).$$

We are interested in evaluating correlation functions, which are the expected values of operator products in the vacuum, or lowest energy state. If we insert a complete set of energy eigenstates $\sum_n |n\rangle \langle n| = 1$ at the ends of our operator, we have

$$\begin{aligned} &\int \mathcal{D}\phi \; \phi_2(x_2^\mu)\phi_1(x_1^\mu) \exp \; i \int_{t_i}^{t_f} d^4x \mathcal{L}(\phi) \\ &= \sum_n \sum_m \langle\phi_f|n\rangle \langle n| \, e^{-it_f E_n} T\left(\phi_H(x_2^\mu)\phi_H(x_1^\mu)\right) e^{it_i E_m} \, |m\rangle \langle m|\phi_i\rangle. \end{aligned}$$

Now we can formally extract the vacuum matrix elements by formally letting t_f approach $+\infty$ with a negative imaginary part, $t_f \to \infty(1 - i\epsilon)$. If we call the lowest state $\langle\Omega|$ and assume its energy is 0, all other states get a contribution $\exp -(E_n\epsilon \cdot \infty) \to 0$, and we extract only the vacuum state from the sum. Similarly we let $t_i \to -\infty(1 - i\epsilon)$. Assume the states $\langle\phi_f|$ and $\langle\phi_i|$ have some overlap with the vacuum state $\langle\Omega|$. Then

$$\begin{aligned} \langle\Omega| \, T\phi_H(x_2)\phi_H(x_1) \, |\Omega\rangle &= \frac{\langle\Omega| \, T\phi_H(x_2)\phi_H(x_1) \, |\Omega\rangle}{\langle\Omega|\Omega\rangle} \\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \; \phi_2(x_2^\mu)\phi_1(x_1^\mu) \exp \; i \int_{-T}^{T} d^4x \mathcal{L}(\phi)}{\int \mathcal{D}\phi \; \exp \; i \int_{-T}^{T} d^4x \mathcal{L}(\phi)}. \end{aligned}$$