Physics 615 — Fall, 2013

Homework #2 — Due: 3 PM, Sept. 20

These problems are all examples of symmetries treated by Noether's theorem. In evaluating $\delta\mathcal{L}$ it is often easier to use the expression

$$\delta\mathcal{L} = \left(\partial_\mu \delta x^\mu\right)\mathcal{L} + \delta\phi_i \frac{\partial\mathcal{L}}{\partial\phi_i} + (\delta\partial_\mu\phi_i)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} + \delta x^\mu \frac{\delta\mathcal{L}}{\delta x^\mu},$$

in the notation of Lecture 4, rather than the expression involving $\mathfrak{d}\phi_i$ and $\dfrac{\partial\mathcal{L}}{\partial x^\mu}$.

1.) Consider the Lagrangian for a set of free real scalar fields $\phi_i, i = 1...N$,

$$\mathcal{L} = \frac{1}{2}\sum_i \partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi^2, \tag{1}$$

where $\phi^2 := \sum_i \phi_i^2$. Consider the internal symmetry transformation

$$\delta x^\mu = 0, \qquad \delta\phi_i = \epsilon A_{ij}\phi_j,$$

where A_{ij} is an antisymmetric real matrix and we are assuming the summation convention that repeated indices are summed over. Show that this is a symmetry by calculating $\delta\mathcal{L}$, and find the conserved current J^μ and the charge Q corresponding to a given infinitesimal ϵA_{ij}. This is a global $SO(N)$ symmetry.

2.) For the case $N = 2$ above, there is only one symmetry generator in $SO(2)$, with $A_{12} = 1 = -A_{21}, A_{11} = A_{22} = 0$. Define ϕ without an index to be the complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, as we did on homework #1. Express $\delta\phi$ and $\delta\phi^\dagger$ in terms of ϵ, ϕ, and $\phi^\dagger$, and what the form of the finite symmetry transformation $\phi \to \phi'$ should be. Express the current and the conserved charge in terms of ϕ and $\phi^\dagger$.

3.) Consider the Lagrangian (1) of problem 1, but now consider symmetry under translations,

$$\delta x^\mu = \epsilon a^\mu(\text{constant}), \qquad \delta\phi_i = 0.$$

Show this is a symmetry, and find the conserved current $J^\mu = a_\nu T^{\nu\mu}$, which gives you the energy momentum, or stress-energy, tensor $T^{\nu\mu}$. Write $T^{\nu\mu}$, $\vec{P}$, and H in terms of the fields ϕ_i.

4.) Again consider the Lagrangian (1) of problem 1, but now for a Lorentz transformation $\delta x^\mu = \epsilon L^\mu{}_\nu x^\nu$, with $L_{\mu\nu} = -L_{\nu\mu}$ as befits the generator of a Lorentz transformation. The current is then[1] $J^\mu = \frac{1}{2} L_{\nu\rho} \mathcal{M}^{\mu\nu\rho}$. Find the expression for $\mathcal{M}^{\mu\nu\rho}$ in terms of the stress-energy tensor. Impose an appropriate permutation symmetry on the indices of $\mathcal{M}$.

5.) Once again consider the Lagrangian (1) of problem 1, but now for a dilation $\delta x^\mu = \epsilon x^\mu$. Can you find a symmetry of this form in general or for some particular m? If so, find the appropriate $\delta\phi$, the current and the charge.

[1] The one-half is inserted because the summation convention includes both L_{12} and L_{21}, though these are equivalent, with only the $L_{\mu\nu}$ with $\mu < \nu$ independent. We therefore also require $\mathcal{M}^{\mu\nu\rho} = -\mathcal{M}^{\mu\rho\nu}$.