Lightning review of groups

We are going to be very concerned with symmetries of our theories, and symmetries form a group. I want to give a lightning review of what we need from Group Theory, though I suspect you will want to learn much more, for example by taking 618.

A group is a set $\mathcal{G}$ of elements g_1, g_2, ..., together with a "multiplication law" $g_1 \odot g_2$ (or simply $g_1 g_2$) such that

a) If $g_1 \in \mathcal{G}$ and $g_2 \in \mathcal{G}$, then $g_1 \odot g_2 \in \mathcal{G}$. This is called closure, the set is closed under group multiplication.

b) For all $g_1, g_2, g_3 \in \mathcal{G}, g_1 \odot (g_2 \odot g_3) = (g_1 \odot g_2) \odot g_3$. This is called associativity.

c) $\exists e \in \mathcal{G}$ such that $\forall g_1 \in \mathcal{G}, e \odot g_1 = g_1 \odot e = g_1$. e is called the identity, and can also be written as 1 or $\mathbb{I}$.

d) For each $g_1 \in \mathcal{G}$, there is a $g_2 \in \mathcal{G}$ such that $g_1 \odot g_2 = g_2 \odot g_1 = e$. g_2 is called the inverse of g_1 and is written g_1^{-1}.

The groups we are interested in are **symmetries**, which are transformations of the degrees of freedom. The multiplication law will always be composition, so $g_1 \odot g_2$ means first apply the transformation g_2 and then apply the transformation g_1. The do-nothing transformation is the identity, and the associative law is automatic for composition.

We are primarily interested in symmetry transformations which act **linearly** on the fields. For example, if we have N scalar fields ϕ_i, we might have invariance under orthogonal transformations

$$g : \phi_i \to \phi'_i = \sum_j \mathcal{O}_{ij} \phi_j,$$

where $\mathcal{O}$ is an orthogonal $N \times N$ real matrix. More generally, as long as the transformation in linear, we will have

$$g : \phi_i \to \phi'_i = \sum_j M_{ij}(g) \phi_j,$$

where $M(g)$ is an $N \times N$ matrix which is a **representation** of the group. That means

$$M(g_1 g_2) = M(g_1) M(g_2).$$

Although there are useful discrete groups of interest, such as that involving parity, time-reversal and charge-conjugation invariance, we are primarily interested in continuous groups, of which rotations are an example. These group elements are described in terms of one or more parameters, such as the angle of the rotation, along with the direction of the rotation axis. A rotation through a finite angle can be thought of as arising from repeated infinitesimal transformations. These infinitesimal transformations differ from the identity by an infinitesimal parameter multiplied by a symmetry generator. An example is the z component of the angular momentum, J_3. A rotation about the z axis can be thought of as $e^{i\theta J_3}$, as an operator, and it acts, for example, on Dirac fields by

$$\psi \to \psi' = e^{i\theta J_3} \psi = e^{i\theta \Sigma_3/2} \psi = e^{-\theta \gamma_1 \gamma_2/2} \psi,$$

where the Dirac field ψ transforms as a 4 dimensional representation (reducible to two 2-dimensional representations) of the rotation group.

The generators span a vector space known as the Lie algebra $\mathfrak{g}$ of the group $\mathcal{G}$. For the rotation group there are three generators J_i and three parameters, so any element of the Lie algebra can be written $\sum_{i=1}^3 \theta_i J_i$, and the group elements are[1] $\exp\left\{ i \sum_{i=1}^3 \theta_i J_i \right\}$.

The rules of group multiplication determine the commutation relations of the generators, which always have the form

$$[J_i, J_j] = i c_{ij}{}^k J_k,$$

where summation over k is understood. The $c_{ij}{}^k$ are known as the structure constants of the algebra. For rotations generated by conventionally normalized angular momentum operators, the structure constants are $c_{ij}{}^k = \epsilon_{ijk}$, where ϵ_{ijk} is the Levi-Civita symbol[2].

Each Lie algebra has a bilinear Killing form $\beta : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{R}$ defined by

$$\beta(L_i, L_j) = -c_{ai}{}^b c_{bj}{}^a.$$

By appropriate choice of the basis L_i for the algebra, the Killing form can be made diagonal. Most of the algebras we will be concerned with are also

[1] Mathematicians generally incorporate the i into the generator, which makes their generators for unitary representations anti-hermitian while ours are hermitian.

[2] It can be defined by the properties that its indices take on the values 1, 2, 3, ϵ_{ijk} is totally antisymmetric under interchange of indices, and $\epsilon_{123} = 1$.

semisimple algebras of compact groups, in which case the basis can be chosen so that $\beta(L_i, L_j) = 2\delta_{ij}$. The Killing form generates the simplest of the Casimir operators, $\sum_{ij} \beta(L_i, L_j) L_i L_j$. More generally, Casimir operators are expressions in terms of the generators which commute with all the generators. The most famous is the $\vec{J}^2$ for the rotation group, the square of the angular momentum.