# Notes on Bessel Functions

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Bessel functions $J_m(x)$ of integral order $m$ may be defined by the generating function

$$g(x,t) := e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n \tag{1}$$

As the generating function is unchanged by $x \to -x, t \to 1/t$, we have $J_{-n}(-x) = J_n(x)$, but it is also unchanged by $x \to -x, t \to -t$, so $J_n(-x) = (-1)^n J_n(x)$, so $J_{-n}(x) = (-1)^n J_n(x)$

Differentiating (1) with respect to $t$ gives

$$\frac{1}{2}x\left(1+\frac{1}{t^2}\right) g(x,t) = \sum_{\mathbb{Z}} n J_n(x) t^{n-1} = \frac{x}{2}\sum_{\mathbb{Z}} J_m \left(t^m + t^{m+2}\right)$$

$$\implies J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x). \tag{2}$$

Differentiating (1) with respect to $x$ gives

$$\frac{1}{2}\left(t-\frac{1}{t}\right) g(x,t) = \sum_{\mathbb{Z}} J_n'(x) t^n = \sum_{\mathbb{Z}} J_m(x) \left(t^{m+1} - t^{m-1}\right)$$

$$\implies J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x). \tag{3}$$

As a special case, $J_0'(x) = -J_1(x)$. (2) $\mp$ (3) gives

$$J_{n\pm1}(x) = \frac{n}{x} J_n(x) \mp J_n'(x). \tag{4}$$

Manipulating Eqs. (4), even if $m$ is not an integer, gives the Bessel equation

$$x^2 \frac{d^2}{dx^2} J_\nu(x) + x\frac{d}{dx} J_\nu(x) + (x^2 - \nu^2) J_\nu(x) = 0 \tag{5}$$

This can also be written

$$u\frac{d}{du} u \frac{d}{du} J_\nu(\alpha u) = (\nu^2 - \alpha^2 u^2) J_\nu(\alpha u)$$

Orthogonality:
Then

$$J_\nu(\alpha u)\frac{d}{du}u\frac{d}{du}J_\nu(\beta u) - J_\nu(\beta u)\frac{d}{du}u\frac{d}{du}J_\nu(\alpha u) = \left(\alpha^2-\beta^2\right)uJ_\nu(\alpha u)J_\nu(\beta u)$$

Integrate from 0 to 1:

$$\begin{aligned}\left(\alpha^2-\beta^2\right)\int_0^1 du\,u\,J_\nu(\alpha u)J_\nu(\beta u) &= \int_0^1 J_\nu(\alpha u)\frac{d}{du}u\frac{d}{du}J_\nu(\beta u) - (\alpha\leftrightarrow\beta)\\ &= \left[uJ_\nu(\alpha u)\frac{d}{du}J_\nu(\beta u)\Big|_0^1 - \int_0^1 u\left(\frac{d}{du}J_\nu(\alpha u)\right)\frac{d}{du}J_\nu(\beta u)\right] - (\alpha\leftrightarrow\beta)\\ &= uJ_\nu(\alpha u)\frac{d}{du}J_\nu(\beta u)\Big|_0^1 - uJ_\nu(\beta u)\frac{d}{du}J_\nu(\alpha u)\Big|_0^1. \qquad (6)\end{aligned}$$

For $\nu \geq 0$ we may assume $J_\nu(0)$ is finite, so the lower endpoint gives zero. If $\alpha$ and $\beta$ are both zeros of $J_\nu$ or both zeros of $J'_\nu$, the upper endpoint also vanishes, so

$$\left.\begin{aligned}\int_0^1 du\,u\,J_\nu(x_{\nu n}u)J_\nu(x_{\nu m}u) = 0\\ \int_0^1 du\,u\,J_\nu(x'_{\nu n}u)J_\nu(x'_{\nu m}u) = 0\end{aligned}\right\}\quad \text{for } m\neq n, \nu\geq 0$$

If we first differentiate (6) with respect to $\alpha$, and then set $\beta=\alpha$, we get

$$2\alpha\int_0^1 du\,u\,J_\nu^2(\alpha u) = u^2J'_\nu(\alpha u)\frac{d}{du}J_\nu(\alpha u)\Big|_0^1 - u\,J_\nu(\alpha u)\frac{d}{du}\left(\frac{u}{\alpha}\frac{d}{du}J_\nu(\alpha u)\right)\Big|_0^1,$$

where we need to be careful that $dJ_\nu(\alpha u)/d\alpha = uJ'_\nu(\alpha) = \frac{u}{\alpha}\frac{d}{du}J_\nu(\alpha u)$. The lower endpoint vanishes. Using the Bessel equation gives

$$2\alpha\int_0^1 du\,u\,J_\nu^2(\alpha u) = \alpha J'^2_\nu(\alpha) + J_\nu(\alpha)\frac{\alpha^2-\nu^2}{\alpha}J_\nu(\alpha).$$

Substitute $J'_\nu(\alpha)\to\frac{\nu}{\alpha}J_\nu(\alpha) - J_{\nu+1}(\alpha)$ to get

$$\begin{aligned}\int_0^1 du\,u\,J_\nu^2(\alpha u) &= \frac{1}{2}\left(\frac{\nu}{\alpha}J_\nu(\alpha) - J_{\nu+1}(\alpha)\right)^2 + \frac{\alpha^2-\nu^2}{2\alpha^2}J_\nu^2(\alpha)\\ &= \frac{1}{2}J_{\nu+1}^2(\alpha) - \frac{\nu}{\alpha}J_\nu(\alpha)J_{\nu+1}(\alpha) + \frac{1}{2}J_\nu^2(\alpha). \qquad (7)\end{aligned}$$

If we set $\alpha$ to a zero of $J_\nu$, the last two terms vanish and

$$\int_0^1 du\, u\, J_\nu^2(x_{\nu\,n}u) = \frac{1}{2} J_{\nu+1}^2(x_{\nu\,n}u). \tag{8}$$

If we set $\alpha$ to $x'_{\nu\,n}$, a zero of $J'_\nu = \frac{\nu}{x'_{\nu\,n}} J_\nu(x'_{\nu\,n}) - J_{\nu+1}(x'_{\nu\,n})$, so $J_{\nu+1}(x'_{\nu\,n}) = \frac{\nu}{x'_{\nu\,n}} J_\nu(x'_{\nu\,n})$, and (7) becomes

$$\int_0^1 du\, u\, J_\nu^2(x'_{\nu\,n}u) = \frac{1}{2}\left[1 - \left(\frac{\nu}{x'_{\nu\,n}}\right)^2\right] J_\nu^2(x'_{\nu\,n}) \tag{9}$$