Physics 613 Lecture 20 April 15, 2014

Gauge Fields

Last time we learned how we could have a fermion multiplet of fields transforming under a gauge symmetry group, and retain the symmetry even under local symmetry transformations, if we introduced a gauge field $A^\mu_a(x)$ for each direction j of the Lie algebra — that is, for each independent infinitesimal symmetry generator. We also found that the field-strength tensor for these gauge fields had, in addition to the $\partial_\mu A_\nu$ terms, terms without derivatives quadratic in the gauge fields. If the group structure constants are f_{abc}, with $[T_a, T_b] = i f_{abc} T_c$, we have

$$F^{\mu\nu}_c = \partial^\mu A^\nu_c - \partial^\nu A^\mu_c - g f_{abc} A^\mu_a A^\nu_b.$$

This F transforms like a representation should, without extra terms proportional to the derivatives of the gauge transformation. From last time we saw

$$\mathcal{F}^{\mu\nu} = e^{ig\vec{\alpha}(x)\cdot\vec{T}} \mathcal{F}^{\mu\nu} e^{-ig\vec{\alpha}(x)\cdot\vec{T}},$$

so under an infinitesimal gauge transformation $\vec{\alpha}$, we have

$$\mathcal{F}^{\mu\nu} \to \mathcal{F}^{\mu\nu} + ig\left[\vec{\alpha}\cdot\vec{T}, \mathcal{F}^{\mu\nu}\right], \qquad \text{or} \qquad F^{\mu\nu}_c \to \mathcal{F}^{\mu\nu}_c - g\sum f_{abc}\alpha_a F^{\mu\nu}_b.$$

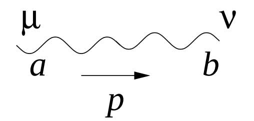
This is the way we expect an *adjoint* representation (octet for SU(3) or spin 1 for SU(2)) to transform. Note that the gauge field itself does not transform that way, as

$$\mathcal{A}^\mu \to +\frac{1}{ig} e^{ig\vec{\alpha}(x)\cdot\vec{T}} \left(\partial^\mu + ig\mathcal{A}\right) e^{-ig\vec{\alpha}(x)\cdot\vec{T}} \Longrightarrow A^\mu_c \to A^\mu_c - \partial^\mu\alpha_c(x) - g f_{abc}\alpha_a A^\mu_b.$$

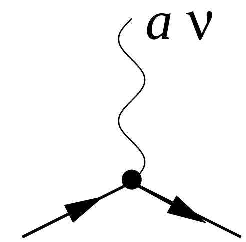
The gauge field picks up an inhomogeneous piece, $-\partial^\mu\alpha_c(x)$, not

What happens when we try to use perturbation theory on our gauge theory? We take, as usual, the unperturbed lagrangian to be only the terms quadratic in the fields, and so the quadratic terms in F, which give cubic and quartic terms in the lagrangian, do not enter into the bare propagators. Thus we have the same ambiguity as we had for electromagnetism, and the propagator for the gauge fields will be

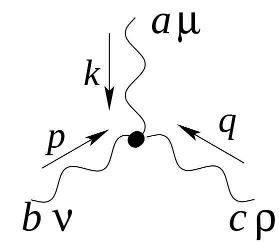
$$\tilde{D}^{\mu\nu}_{F,ab}(k) = i\frac{-g^{\mu\nu} + (1-\xi)k^\mu k^\nu/k^2}{k^2 + i\epsilon}\delta_{ab}.$$

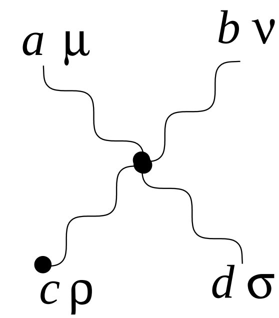


The lines on the Feynman diagram will have a Lorentz index μ and an adjoint color index j on each end.



The spinor propagator is unchanged, except that there will be a fundamental color index f (which takes on N equal values for SU(N)) on each end. The interaction of the spinor with the gauge field, $-\frac{1}{2}g\bar{\psi}_f\tau^a_{fh}\gamma^\mu\psi_h$ acts as a matrix γ^μ in spinor space and τ^a on the fundamental color index of the spinor product of factors in the Feynman diagram.





But an important change in going to non-Abelian gauge fields is that now there are pure gauge field interaction terms coming from the parts of $-\frac{1}{4}F^{\mu\nu}_a F_{j\,\mu\nu}$ where one or both F's contribute their quadratic terms. We have in $\mathcal{L}$ a term $\frac{1}{2}g f_{abc}\partial_\mu A_{c,\nu} A^\mu_a A^\nu_b$ and another one $-\frac{g^2}{4}A_{a,\mu}A_{b,\nu}A^\mu_c A^\nu_d \sum_e f_{abe}f_{cde}$. As the incoming lines can hit any of the A's, there will be six terms in the three-gauge vertex, with a vertex $g f_{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$, and the four-gauge vertex is given by $-ig^2\sum_e f_{abe}f_{cde}g^{\mu\rho}g^{\nu\sigma}$ plus five other permutations of the 4 particles (there are 24 permutations overall, but each of the six terms we have described, each coming from which pairs get contracted, gets equal contributions from four of these permutations.

Effects of 3- and 4- gauge particle interactions

Though we have based our gauge theory on the electromagnetic gauge theory with which we are very familiar, the fact that we now have three and four gauge particle interaction terms in the Lagrangian has some profound effects. Perhaps we should point out that in electromagnetism, without other particles the photons would be noninteracting particles. Photons couple to charge, but are themselves uncharged. The gauge particles, however, couple to all particles which transform nontrivially under the group, and they themselves transform[1] as the adjoint representation of the group.

One of the important consequences of this has to do with the renormalization group, or how the effective coupling constants depend on where, in momentum, the renormalization is done. We did not consider this in QED because the natural point to define the renormalization is the electron coupling at zero q^2, the physically measured charge, and the measured nonzero electron mass. If we are discussing QCD, the non-Abelian gauge theory of color SU(3), the fundamental particles, quarks and gluons, are confined and do not give us natural points in momentum space at which to specify our physical parameters.

Though we did not discuss it here[2], if one examines the photon propagator at large spacelike momentum, including loop effects, we find that the coupling to the electron acts with an effective coupling $\alpha(q^2)$ which is not the constant 1/137 but instead increases at high momentum. This is understood as saying that at high momentum we are seeing more of the bare charge, which is screened by vacuum polarization when we look with lower q^2 with less spatial resolution. So electromagnetism gets stronger in the ultraviolet.

The electromagnetic coupling is quite weak at most scales at which we wish to know the interactions, so this does not bother us, but the strong interactions are so strong at the MeV or GeV scale that perturbation theory does not help us calculate physical properties. For this reason, in the '60's one despaired of being able to explore the strong interactions using field theory at all, until it was realized that the effective coupling constant might conceivably decrease instead of increasing with energy, and so perhaps high energy processes could be understood using perturbation theory. Which way the coupling grows with energy is determined by something called the beta function, which can be calculated from loop graphs in the theory. At the time, all known theories behaved like electromagnetism, the wrong way. But in 1973, Gross and Wilczek at Princeton and Politzer at Harvard calculated the beta function for non-Abelian gauge theories[3] and found the coupling constant ran the other way, getting weaker at high energy (asymptotic freedom) and stronger at low energy (infrared slavery). In fact, we now understand the

[1] Here we are not considering the inhomogeneous term from the local character of the gauge — gauge particles are transformed even by global gauge transformations with constant $\vec{\alpha}$.

[2] I did in 615.

[3] And thereby won a Nobel prize.

long-distance behavior to be so strong that color is confined — the potential between oppositely colored particles grows infinitely as they are separated to infinity.

Another issue complicated by the self-coupling of gauge particles is that maintaining the Ward identity becomes more difficult. In QED, the longitudinal and timelike polarizations conspired to cancel each other in all appropriate contexts, but this does not happen in non-Abelian theory. Instead, one needs to introduce other fictitious particles called ghosts. These particles never appear as external particles, but only circulate in loops appropriately to cancel unwanted parts of the gluon propagator. They are scalar particles but obey fermi statistics, with anticommuting fields. If you want to know more about this, take 616 (I think).