Physics 613 Lecture 19 April 10, 2014

Chiral Symmetry; Non-Abelian Gauge Symmetry

The isospin or flavor SU(3) symmetry we are discussing deal with degrees of freedom that have nothing to do with spacetime, but describe a sort of rotation in a completely independent space, and the particle fields at each point in space-time transform identically, so a quark field $\psi_{j,a}(x)$ with flavor index j, spinor index a and space-time point x^μ transforms by $\psi_{j,a}(x) \to M_{jk}(g)\psi_{k,a}(x)$ where the representation matrix M is completely independent of a and x^μ. I will call this an internal symmetry.

Chiral Symmetry

We have seen that for charged fields, the lagrangian is invariant under a global phase transformation $\psi_a(x) \to e^{i\alpha}\psi_a(x)$, for real constant α, so this is a form of internal symmetry, and results in a conserved current (which turns out to be the electric current) due to Noether's theorem. We see that the standard free Dirac Lagrangian density $\mathcal{L} = i\bar\psi \not\partial\psi - m\bar\psi\psi$ is invariant, because $\bar\psi = \psi^\dagger\gamma^0 \to e^{-i\alpha}\bar\psi = \bar\psi e^{-i\alpha}$. Let us consider a similar transformation which rotates the two helicity states of the particle in opposite directions, $\psi_a(x) \to e^{i\alpha\gamma_5}\psi_a(x)$. We still have $\psi^\dagger \to \psi^\dagger e^{-i\alpha\gamma_5}$, as γ_5 is hermitean, but as γ_5 anticommutes with γ^0, $\bar\psi \to \bar\psi e^{+i\alpha\gamma_5}$, and $\bar\psi\psi$ is not invariant. γ_5 does commute with $\gamma^0 \not\partial$, so the first term in $\mathcal{L}$ is invariant under this *chiral transformation*. So if we have a **massless** free dirac particle, we have chiral symmetry and a conserved current and conserved quantity Q_5, which is the number of right-handed fermions minus the number of left-handed ones. As a consequence, Q_5 anticommutes with the parity operator.

Note that if our massless dirac particle interacts with the electromagnetic field in the usual way, with $\mathcal{L}_I = q\bar\psi\gamma_\mu\psi A^\mu$, this will be invariant under chiral transformations provided A^μ is unchanged.

Do we have any massless fermions to apply this to? Perhaps not in the spectrum of particles we observe, although neutrinos come close. In high energy electron scattering we can often consider the mass of the electron to be negligible, and the consequence of the approximate chiral symmetry is that the helicity of the electron is nearly always conserved, with helicity-flip amplitudes going away at high energy/momentum.

We have also come to consider the u and d quark to be very light. Early quark models thought their masses would be roughly $m_N/3$, with m_N the mass of the nucleon, proton or neutron, but these *constituent* masses are now not considered to be essential, and one considers *current quark masses* which are 4-10 MeV, quite a bit smaller that the masses of objects which contain them.

Of course we also have the (approximate) isotopic spin symmetry $q \to q' = e^{-i\vec{\alpha}\cdot\vec{\tau}/2}q$. Noether tells us we have conserved currents

$$\hat{J}^\mu_j = \frac{1}{2}\bar{q}\tau_j\,\gamma^\mu q, \qquad \text{with conserved isocharges}$$
$$\hat{Q}_j = \hat{T}^{\frac{1}{2}}_j = \int \hat{J}^0_j d^3x = \int q^\dagger(x)\frac{\tau_j}{2}q(x)$$

As these conserved isocharges generate the isorotations, they have SU(2) commutation relations,

$$[\hat{Q}_j, \hat{Q}_k] = i\epsilon_{jk\ell}\hat{Q}_\ell. \tag{1}$$

If our quarks are massless, we can combine isospin and chiral symmetry and consider

$$q \to e^{-i\vec{\alpha}\cdot\vec{\tau}\,\gamma_5/2}\,q.$$

This is not quite an internal symmetry, because the γ_5 acts on the spinor indices and converts one quark into another with a possibly different spin. If this were a symmetry of our theory, its conserved current would be

$$\hat{J}^\mu_{j,5} = \frac{1}{2}\bar{q}\tau_j\,\gamma^\mu\gamma_5 q, \qquad \text{with conserved charge}$$
$$\hat{Q}_{j,5} = \hat{T}^{\frac{1}{2}}_{j,5} = \int \hat{J}^0_{j,5} d^3x = \int q^\dagger(x)\gamma_5\frac{\tau_j}{2}q(x)$$

The operator $\bar{q}\gamma_5\vec{\tau}q$ is a pseudoscalar, isotopic spin 1 object, and if we let it operate on the vacuum it would create a quark-antiquark pair suitable for a pion. If we dotted this into a pion field, it would provide a suitable coupling of a quark with a pion. As it is not chiral invariant, we need the pion field to change under chiral transformations.

We will see later, if we have time, that the low mass of the pion can be connected to the breaking of the chiral symmetry of the u and d quarks.

Just as something recognizable from things we have seen earlier, and as a foretaste of discussions in Chapter 18, note that the $\hat{Q}_j$ and $\hat{Q}_{j,5}$ form a six-dimensional Lie algebra with commutation relations (1) together with

$$[\hat{Q}_j, \hat{Q}_{k,5}] = i\epsilon_{jk\ell}\hat{Q}_{\ell,5}, \tag{2}$$
$$[\hat{Q}_{j,5}, \hat{Q}_{k,5}] = i\epsilon_{jk\ell}\hat{Q}_{\ell}. \tag{3}$$

This should look rather familiar, from a discussion of $\vec{J}$ and $\vec{K}$ from the Lorentz group, except there the $[K, K]$ commutator had a negative sign. That required complexifying $J \pm iK$ to find finite dimensional representations of the Lorentz group, but here things are simpler. If we define $\hat{Q}_{j,\pm} = \frac{1}{2}\left(\hat{Q}_j \pm \hat{Q}_{j,5}\right)$, and rename $\hat{Q}_{j,+} \to \hat{Q}_{j,R}$ and $\hat{Q}_{j,-} \to \hat{Q}_{j,L}$, we have right and left handed SU(2) algebras which commute with each other, and furthermore are given by

$$\hat{Q}_{j,\pm} = \int d^3x\, q^\dagger(x)\, \frac{1 \pm \gamma_5}{2}\, \frac{\tau_j}{2}\, q(x) = \int d^3x\, q_\pm^\dagger(x)\, \frac{\tau_j}{2}\, q_\pm(x),$$

where $q_\pm = \dfrac{1 \pm \gamma_5}{2}\, q$ are the right and left handed helicity components of the quark field.

The idea that we might have separate symmetries on the left and right handed components will prove central to our standard model, but the internal symmetry group will not be hadronic isospin but weak isospin. Which brings us to:

Non-Abelian Gauge Theories

Yang and Mills (1954) thought it inconsistent that we should have a global isotopic spin symmetry which asserted that protonness and neutronness were just arbitrary directions chosen in isotopic spin space, which one could choose arbitrarily at one point, but not independently at other space-time points. Relativity, after all, tells us that all physics is local. So they asked the question of whether it would be possible to have a theory for an isospin doublet $\psi(x)$ invariant under local isospin transformations

$$\psi(x) \to \psi'(x) = e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\, \psi(x).$$

We have scaled the parameters $\vec{\alpha}$ by a coupling constant g in the same way as we have θ with the electric charge in gauge invariance for electromagnetism. For terms in the lagrangian that do not have derivatives, global invariance and local invariance are indistinguishable. But for

$$\begin{aligned} i\bar{\psi}\,\partial\!\!\!/\psi &\rightarrow i\bar{\psi}e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\gamma^\mu\partial_\mu\left(e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\,\psi\right) \\ &= i\bar{\psi}e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\gamma^\mu\left(\partial_\mu e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\right)\psi + i\bar{\psi}\,\partial\!\!\!/\psi, \end{aligned}$$

we get a change

$$i\bar{\psi}\left[e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\partial_\mu e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\right]\gamma^\mu\psi,$$

which does not vanish if $\alpha(x)$ has a nonzero derivative. For an Abelian theory, the derivative of the exponential is just $\frac{1}{2}i\vec{\tau}\cdot\partial_\mu\vec{\alpha}$ times the exponential, which cancels the one from $\bar{\psi}$, and the change is just $\frac{1}{2}\,ig(\partial_\mu\vec{\alpha})\cdot\bar{\psi}\gamma^\mu\vec{\tau}\psi$, which is compensated in gauge transformations by

$$A_\mu \rightarrow A_\mu - ig\partial_\mu\chi.$$

If we want to do the same for our SU(2) isospin group, we need a gauge field with components which correspond to all the components of $\vec{\alpha}$. As these multiply the generators of the group, the Lie algebra, we need a gauge field which takes on not real values, but instead values in the Lie algebra,

$$\mathcal{A}_\mu = \sum_j A_j\hat{T}_j = \vec{A}_\mu\cdot\vec{\tau}/2.$$

This suggests using a covariant derivative

$$D_\mu(\mathcal{A}) := \partial_\mu + ig\mathcal{A}_\mu$$

in place of ∂_μ in the kinetic term of the dirac particle. We want that

$$\begin{aligned} \bar{\psi}\gamma^\mu D_\mu(\mathcal{A})\psi &\rightarrow \bar{\psi}'\gamma^\mu D_\mu(\mathcal{A}')\psi' = \bar{\psi}e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\gamma^\mu D_\mu(\mathcal{A}')\left(e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\,\psi\right) \\ &= \bar{\psi}e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\gamma^\mu\left[D_\mu(\mathcal{A}), e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\right]\psi \\ &\quad +ig\bar{\psi}e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\gamma^\mu(\mathcal{A}'-\mathcal{A})e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\,\psi \\ &\quad +\bar{\psi}\gamma^\mu D_\mu(\mathcal{A})\psi \end{aligned}$$

The last term is what we want, so the other two must cancel, which it will if we set

$$ig(\mathcal{A}'-\mathcal{A})e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2} = -\left[D_\mu(\mathcal{A}), e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}\right]$$

or

$$\mathcal{A}' = \frac{1}{ig}e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2}D_\mu(\mathcal{A})e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}.$$

For an infinitesimal gauge transformation α, this gives

$$\mathcal{A}' = \mathcal{A} - \partial_\mu \vec{\alpha}(x) \cdot \vec{\tau}/2 + ig[\vec{\alpha}(x) \cdot \vec{\tau}/2, \mathcal{A}_\mu],$$

$$\text{or} \qquad A'_{\ell,\mu} = A_{\ell,\mu} - \partial_\mu \alpha_\ell - g f_{jk\ell} \alpha_j A_k$$

.

The Field Strength

So we have found how to make a lagrangian with matter fields that transform under a symmetry group independently at each point in space-time, but at the expense of having to introduce a vector field which takes values in the Lie algebra of the group. But the new field needs to have its part of the lagrangian too, in particular the kinetic energy term.

For electromagnetism we know that term is $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The best way to see how to generalize this is to note that the covariant derivative is analogous to that of general relativity, where it tells you how to parallel transport a vector, and the curvature tensor, which is the commutator of two covariant derivatives, when integrated around a closed path, tells how much a parallel-transported vector has rotated when it comes back to where it started.

We note that for the electromagnetic field, we could also write

$$F_{\mu\nu} = \frac{-i}{g}\,[D_\mu, D_\nu] \tag{4}$$

which will also turn out to be the correct expression for the non-Abelian theory, with

$$\mathcal{F}_{\mu\nu} = \frac{-i}{g}\,[\partial_\mu + ig\mathcal{A}_\mu, \partial_\nu + ig\mathcal{A}_\nu] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + ig[\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{5}$$

or, in terms of components in the Lie algebra space,

$$F_{j,\mu\nu} = \partial_\mu A_{j,\nu} - \partial_\nu A_{j,\mu} - g f_{k\ell j} A_{k,\mu} A_{\ell,\nu}.$$

We need to have a term similar to the $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ of electromagnetism, but we must insure that it is invariant under the gauge symmetry. Observe that the way we have defined the action of the gauge transformation on $\mathcal{A}_\mu$, we have $D_\mu \to e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2} D_\mu e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}$, so

$$\mathcal{F}_{\mu\nu} = e^{ig\vec{\alpha}(x)\cdot\vec{\tau}/2} \mathcal{F}_{\mu\nu} e^{-ig\vec{\alpha}(x)\cdot\vec{\tau}/2}.$$

As the gauge fields and their field strengths live in the vector space of symmetry generators, the scale of the component fields $F_{j,\mu\nu}$ is determined by some means of normalizing the basis vectors of this space, the $\hat{T}_j$. For SU(2) the convention is $\vec{\tau}/2$, where the τ's are the Pauli vectors renamed only to remind us that this is in isospin space rather than real space. The mathematicians have given us what we need to extract an invariant, by defining the *Killing form* $\beta(\hat{T}_j, \hat{T}_k)$ which gives the first Casimir operator $\sum_{jk} \beta(\hat{T}_j, \hat{T}_k)\hat{T}_j\hat{T}_k$. For most of the algebras we care about, the $\hat{T}_j$ can be chosen so that $\beta(\hat{T}_j, \hat{T}_k) = 2\delta_{jk}$, which we will do. This is familiar from rotations, where we get $\sum_j L_j^2$ as the Casimir operator invariant under rotations. Now, because the exponentials commute with the Casimir operator, $\beta\left(\mathcal{F}^{\mu\nu}, \mathcal{F}_{\mu\nu}\right) = 2\sum_j F_j^{\mu\nu} F_{j,\mu\nu}$ is an invariant, so we take

$$\mathcal{L} = -\frac{1}{4}\sum_j F_j^{\mu\nu} F_{j,\mu\nu} + i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi$$

to be our lagrangian density.